Int J Thermophys (2015) 36:692–700
DOI 10.1007/s10765-015-1848-9

Concentric Circular Strips Model of the Transient Plane Source-Sensor

Svetozár Malinarič · Peter Dieška

Received: 8 September 2014 / Accepted: 29 January 2015 / Published online: 14 February 2015

Abstract The transient plane heat source (TPS) method is an experimental technique for measurement of the thermal conductivity and diffusivity of solid materials. A complete model based on the concentric circular strips structure of the TPS-sensor has been developed. Rings, circles, and disk models were derived as special cases of the strips, and all four models were compared. Both strips and circles models gave nearly the same results and are recommended for TPS measurement evaluation. Moreover, there was shown that the rings and circles models are incorrect because they lead to an infinite temperature, which can be obviated by using the modified shape functions. In addition, a new design of the electrical bridge was given and the stability of the input heat power was verified by measurements on polymethylmetacrylate.

Keywords Concentric circular strips model · Polymethylmetacrylate · Thermal conductivity · Thermal diffusivity · Transient plane heat source method

1 Introduction

In the light of the energy crisis, the use of better thermal insulation materials is becoming more and more important. Knowledge of thermal conductivity and its precise measurement is crucial for a wide range of applications [1]. There are mainly two approaches for thermal-conductivity measurements: steady-state and transient-

S. Malinarič (✉)
Department of Physics, Constantine the Philosopher University,
Trieda Andreja Hlinku 1, 94974 Nitra, Slovak Republic
e-mail: smalinaric@ukf.sk

P. Dieška
Institute of Nuclear and Physical Engineering, Faculty of Electrical Engineering and Informatics,
Slovak University of Technology, Ilkovičova 3, 81219 Bratislava, Slovak Republic

state. The former uses a steady-state temperature field inside the specimen, which requires long-time measurements. The latter is characterized by short-time measurements, small specimens, and the possibility of obtaining the thermal diffusivity but with complex measurement evaluation [2].

Transient methods utilize the heat source and thermometer, both placed inside the specimen. The experiment consists in measuring the temperature response to the input heat flux with a steady-state initial condition. The evaluation is based on determining the thermal conductivity and diffusivity by fitting the temperature function to the temperature response. In the transient plane heat source (TPS) method, a TPS-sensor serves simultaneously as the heat source and thermometer [3,4]. The sensor is composed of a double spiral strip made from nickel and tightly sandwiched between two specimen plates. The input heat power in the form of the stepwise function is produced by the passage of an electrical current through the sensor.

The temperature function is a solution of the heat equation with boundary and initial conditions corresponding to the experimental arrangement. An essential feature of the solution is the model of the TPS-sensor. Gustafsson [3] gave a model of a certain number of concentric ring sources (rings model) and a model with infinitely small openings between circular strips (disk model). However, in reality the width of the strips equals the width of the gaps. The aim of this study is to create the model very close to this sensor structure. In addition, the integral in [3] results in a singularity and cannot be computed. This is the main reason for the analysis of the TPS-sensor models in this study.

2 Theoretical Model

The theoretical model of the TPS method is created by the following conditions:

- The sensor consists of a set of concentric circular strips.
- The thickness and heat capacity of the sensor are negligible.
- There is no thermal resistance between the sensor and specimen.
- The specimen is infinite in all directions.
- The input power in the sensor is stepwise.

The temperature field of the heated circle is a solution of the heat equation,

$$\frac{\partial T}{\partial t} - k\,\Delta T = \frac{w}{\rho c}, \tag{1}$$

where k, ρ, and c are the thermal diffusivity, density, and mass heat capacity of the surroundings, respectively, and the input heat power density is

$$w(t, \vec{\boldsymbol{r}}) = \frac{P}{2\pi R}\delta(r - R)\delta(z)1(t), \tag{2}$$

where 1() is the Heaviside step function, $\delta()$ is the Dirac delta function, P is the input heat power, R is the radius of the circle, and r, φ, and z are the cylindrical coordinates of the vector $\vec{\boldsymbol{r}}$. The initial and boundary conditions are

$$T|_{t=0} = 0 \qquad T|_{|\vec{\boldsymbol{r}}|\to\infty} = 0. \tag{3}$$

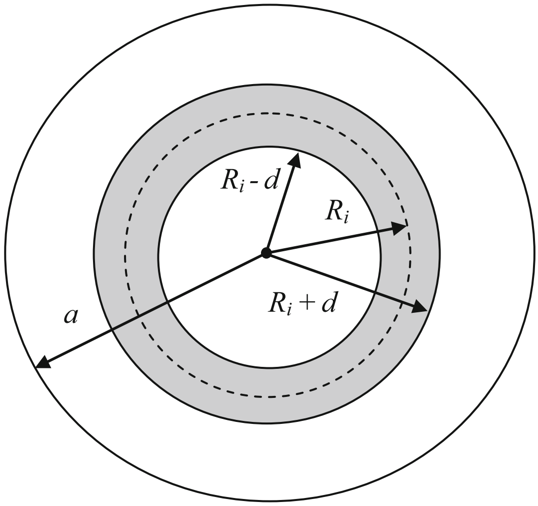

Fig. 1 Model of the TPS-sensor as a set of concentric strips where a is the radius of the sensor and $i = 1 \ldots N$

The solution of Eq. 1 will be

$$T(t, \vec{\boldsymbol{r}}) = \int_0^\infty \mathrm{d}t' \int \mathrm{d}\vec{\boldsymbol{r}}' G(t, \vec{\boldsymbol{r}}, t', \vec{\boldsymbol{r}}') \frac{w(t, \vec{r})}{\rho c}$$
$$= \frac{P}{\rho c} \pi^{-5/2} \int_0^\pi \mathrm{d}\varphi \int_0^t \mathrm{d}t' \left(4kt'\right)^{-3/2} \mathrm{e}^{-\frac{r^2+R^2+2rR\cos\varphi+z^2}{4kt'}}, \tag{4}$$

where we used the Green's function [5]. $\vec{\boldsymbol{r}}'$, t' represent the place and time of the source (heated point) and $\vec{\boldsymbol{r}}$, t represent the temperature field (measured point). The temperature field of the jth strip in the plane $z = 0$ will be obtained by the superposition as

$$T_j(t, r) = \frac{P_j \pi^{-5/2}}{\rho c \ln\left(\frac{R_j+d}{R_j-d}\right)} \int_{R_j-d}^{R_j+d} \frac{\mathrm{d}R}{R} \int_0^\pi \mathrm{d}\varphi \int_0^t \mathrm{d}t' \left(4kt'\right)^{-3/2} \mathrm{e}^{-\frac{r^2+R^2+2rR\cos\varphi}{4kt'}}, \tag{5}$$

where P_j is the power input into the jth strip defined in Fig. 1. The mean temperature of the ith strip caused by the heated jth strip is

$$\overline{T}_{ij} = \frac{1}{2R_i d} \int_{R_i-d}^{R_i+d} r \mathrm{d}r T_j\,(t, r). \tag{6}$$

The mean temperature of the ith strip caused by all heated strips N is

$$\overline{T}_i = \sum_{j=1}^{N} \overline{T}_{ij}. \tag{7}$$

The mean temperature of all strips (temperature function) will be obtained as the weighted mean and using Eqs. 5–7, we have

$$\overline{T}\,(t) = \frac{\sum\limits_{i=1}^{N} \overline{T}_i R_i}{\sum\limits_{i=1}^{N} R_i} = \frac{P}{\pi^{\frac{3}{2}} \lambda a} D\,(\tau)\,, \tag{8}$$

where P is the total input heat power into the sensor, $\lambda = k\rho c$ is the thermal conductivity, a is the radius of the sensor, and $\tau = \sqrt{kt}/a$. The shape function [6] has the form,

$$D\,(\tau) = \frac{a}{8\sqrt{\pi}d\left(\sum\limits_{i=1}^{N} R_i\right)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{R_j}{\ln\left(\frac{R_j+d}{R_j-d}\right)} \times \int\limits_{R_i-d}^{R_i+d} r\mathrm{d}r \int\limits_{R_j-d}^{R_j+d} \frac{\mathrm{d}R}{R} \int\limits_{0}^{\pi} \mathrm{d}\varphi \frac{\operatorname{erfc}\left(\frac{1}{2a\tau}\sqrt{r^2 + R^2 + 2rR\cos\varphi}\right)}{\sqrt{r^2 + R^2 + 2rR\cos\varphi}}, \tag{9}$$

where erfc is the complementary error function [5]. This formula is ready for numerical evaluation. In order to compare this theory with that in [3] according to the correspondence principle, the shape function will be rewritten as follows:

$$D\,(\tau) = \frac{1}{8d\left(\sum\limits_{i=1}^{N} R_i\right)^2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{R_j}{\ln\left(\frac{R_j+d}{R_j-d}\right)} \times \int\limits_{R_i-d}^{R_i+d} r\mathrm{d}r \int\limits_{R_j-d}^{R_j+d} \frac{\mathrm{d}R}{R} \int\limits_{0}^{\tau} \frac{\mathrm{d}\sigma}{\sigma^2} \mathrm{e}^{-\frac{r^2+R^2}{4a^2\sigma^2}} I_0\left(\frac{rR}{2a^2\sigma^2}\right), \tag{10}$$

where I_0 is the modified Bessel function. The strips structure in Fig. 1 is defined by

$$R_i = (4i - 1)\,d, \quad a = 4Nd \tag{11}$$

For very narrow strips $d \ll R_i$, Eq. 10 becomes

$$D(\tau) = \frac{1}{N^2(N+1-2B)^2} \sum_{i=1}^{N} (i-B) \sum_{j=1}^{N} (j-B) \times \int_0^\tau \frac{d\sigma}{\sigma^2} e^{-\frac{(i-B)^2+(j-B)^2}{4N^2\sigma^2}} I_0\left(\frac{(i-B)(j-B)}{2N^2\sigma^2}\right) \quad (12)$$

This is the shape function for the sensor consisting of a certain number of concentric rings. The expression slightly differs from Eq. 23 in [3] which corresponds to the sensor structure as $R_i = 4id$. To distinguish between both models, Eq. 12 will be referred to as the circles model for $B = 1/4$ and as the rings model for $B = 0$. Here it should be emphasized that from a purely mathematical point of view, Eq. 12 is incorrect, because σ causes the singularity in zero. It could be shown for $i = j$ as follows:

$$\int_0^\tau \frac{d\sigma}{\sigma^2} e^{-\frac{(i-B)^2+(j-B)^2}{4N^2\sigma^2}} I_0\left(\frac{(i-B)(j-B)}{2N^2\sigma^2}\right) = \int_0^\tau \frac{d\sigma}{\sigma^2} e^{-\frac{\alpha}{\sigma^2}} I_0\left(\frac{\alpha}{\sigma^2}\right) = \frac{1}{\sqrt{2\pi\alpha}} [\ln|\sigma|]_0^\tau \to \infty, \quad (13)$$

where the asymptotic formula $I_0(x) \approx e^x/\sqrt{2\pi x}$ for $x \gg 1$ has been used. Similarly, we can get the shape function for the disk model of the TPS-sensor with infinitely small openings between strips so that

$$d \to 0, \quad N \to \infty, \quad a = 2Nd \quad (14)$$

and Eq. 10 becomes

$$D(\tau) = \int_0^1 u\,du \int_0^1 v\,dv \int_0^\tau \frac{d\sigma}{\sigma^2} e^{-\frac{u^2+v^2}{4\sigma^2}} I_0\left(\frac{uv}{2\sigma^2}\right). \quad (15)$$

3 Numerical Evaluation

In order to investigate the problem with the singularity in Eq. 12, we computed the derivatives $D'(\tau)$ of all shape functions for $N = 16$ and plotted them in Fig. 2. As τ tends to zero, the curve for the strips model (Eq. 9) approaches 1.939 and the values for the disk model (Eq. 15) go to 1. This can be verified by the following consideration. For $\tau \to 0$, the strips model corresponds to one-dimensional heat flow into an infinite

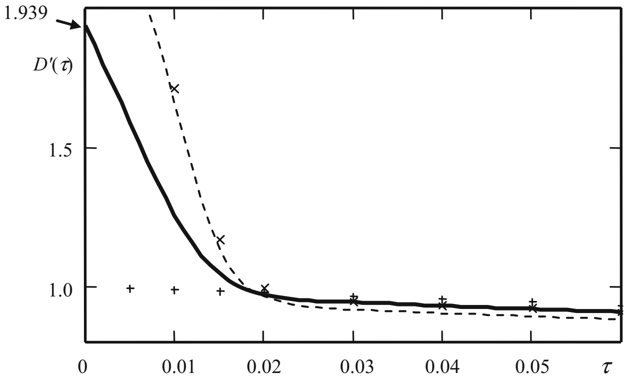

Fig. 2 Plots of the derivatives of the shape functions $D(\tau)$ for strips model (solid line), rings model (dashed line), circles model (x), and disk model (+)

medium [7] with the temperature function,

$$T(t) = \frac{q}{\lambda}\sqrt{\frac{kt}{\pi}}, \tag{16}$$

where q is the power per unit area dissipated by the sensor,

$$q = \frac{P}{S} = \frac{P}{\pi a^2}\frac{4N}{2N+1}, \tag{17}$$

Using Eqs. 8, 16, and 17, the shape function becomes

$$D(\tau) = \frac{4N}{2N+1}\tau = 1.939\tau. \tag{18}$$

Similarly, for the disk model we have $D(\tau) = \tau$. For circles and rings models, $S = 0$ and thus $D(\tau) \to \infty$. As all four plots in Fig. 2 are very similar for $\tau > 0.03$, there is a possibility to use them in experiment evaluation by constructing the modified shape functions,

$$D_{\mathrm{m}}(\tau) = \int_{0.03}^{\tau} D'(\sigma)\,\mathrm{d}\sigma. \tag{19}$$

Table 1 shows the shape functions of the strips and rings models and their relative differences for the τ-range from 0.2 to 1. For reasons mentioned above, the modified shape function of the rings model had to be used and also the function could not be evaluated in zero. However, the differences presented in the table do not show how they influence the results of the thermal conductivity and diffusivity measurements.

It is here important to remember that the TPS-sensor is a double spiral of a strip source with a certain heat capacity. Although the strips model represents a more precise

Table 1 Comparison of the shape functions: s—strips model, r—rings model

τ	D_{s}	D_{r}	$\frac{D_{\mathrm{s}}}{D_{\mathrm{r}}}-1$ (%)
0.2	0.180	0.176	2.3
0.4	0.311	0.304	2.3
0.6	0.405	0.397	2.2
0.8	0.471	0.462	2.1
1.0	0.518	0.508	2.0

solution than the rings model for very short times, it does not consider the heat capacity of the sensor. This causes the non-zero rise time of the heating power, and its influence is reduced by using a time correction t_{c} and removing the deviating points at the beginning of the transient [4]. The best solution of this problem is based on numerical compensation of the input heat power variation [8].

The thermophysical parameters k and λ can be determined by fitting predicted values of the mean temperature of the sensor,

$$\overline{T}(t) = A + \frac{P}{\pi^{3/2} a\lambda} D\left(\frac{\sqrt{k(t-t_{\mathrm{c}})}}{a}\right), \tag{20}$$

to the measured points $[t_i, T_i]$ where A is a nuisance parameter. The thermal diffusivity k and time correction t_{c} will be iterated until the correlation coefficient of the dependence of T_i versus $D(t_i)$ reaches its maximum and λ is determined from the slope of this line. In Eq. 20 the modified shape function can be used under the assumption that $\tau > 0.03$.

4 Experiment

The experiment was included in this paper in order to show the influence of the applied models on the results of thermal conductivity and diffusivity measurements. The resistance of the sensor is measured by using a two-channel nanovoltmeter and the bridge illustrated in Fig. 3. At first the time dependence of the voltage across the bridge u_1 during the transient is measured. Before the heating current is turned off, the voltage of the source u_2 and both voltages U_1 and U_2 in the two-channel mode are measured. The sensor resistance can be calculated as follows:

$$R(t) = (R_{\mathrm{c}} + R_{\mathrm{l}}) \frac{U_a + u_1(t)}{U_b - u_1(t)} - R_{\mathrm{l}}, \tag{21}$$

where R_{c} is a constant resistor and R_{l} is the resistance of one sensor lead which can also be determined from the measurements as

$$R_{\mathrm{l}} = \frac{R_{\mathrm{c}}}{2} \frac{u_2 - U_1 - U_2}{U_2}. \tag{22}$$

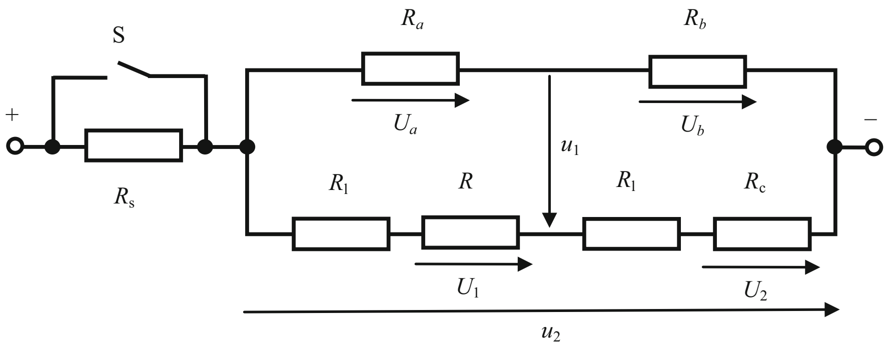

Fig. 3 Bridge for sensor resistance R measurement using two-channel nanovoltmeter. R_a, R_b, and R_c are constant resistors, and R_l is the resistance of the sensor lead. S is the switch for starting the experiment

Table 2 Results of the measurement with various values of input heat power, evaluated using the strips model

P (mW)	λ (W·m^{-1}·K^{-1})	a (mm^2·s^{-1})
20	0.209	0.117
60	0.208	0.116
120	0.208	0.116

As we measure only at the laboratory temperature, the resistance of the sensor changes less then 3 %. So the value of the constant resistor R_c can be set very close to the sensor resistance R. This is necessary for keeping the power input constant during the measurement [4]. The other advantages of the experimental arrangement in Fig. 3 are described in detail in [9].

The measurements were performed at the laboratory temperature by using the sensor Hot Disk AB Type 5501 with a number of rings (16), a radius of 6.4 mm, a resistance of about 13 Ω, and a temperature coefficient of resistivity of 0.0048 K^{-1}. The specimens made from polymethylmetacrylate (PMMA) had a cylindrical shape with a diameter of 30 mm and a thickness of 9 mm. The measurements were performed at various values of the input heat power up to 120 mW, which corresponds to the total sensor temperature increase in the experiment of 5.9 K. Experimental details and typical graph of residuals, which indicate the quality of the measurement, were presented in [9].

Table 2 shows that the results of the thermal conductivity and diffusivity measurements are not dependent on the input heat power. This can be considered as evidence that the input power is constant during the experiment. Hence, the measurements can be done at higher values of the input heat power to achieve higher values of the signal-to-noise ratio.

Table 3 shows the results of one measurement, but evaluated using all four methods analyzed in this paper. The disk, circles, and rings models are compared to the strips model using the relative differences.

Table 3 Relative differences between the results of the measurement evaluated using various models. Three models are compared to the strips model

Model	Disk (%)	Circles (%)	Rings (%)
λ	2.0	0.2	−1.5
a	−2.5	0.0	3.3

Three models are compared to the strips model

5 Summary

A new model of the TPS-sensor consists of a set of N concentric circular strips as illustrated in Fig. 1. The temperature function is the solution of the heat equation and is expressed by means of the shape function in two forms (Eqs. 9 and 10). The latter was used to derive the shape function for the circles and rings models (Eq. 12) for the case when the strips are infinitely narrow. Numerical evaluation and experiment showed that the circles model gives nearly the same results as the strip one, and both should be recommended for experiment evaluation. Rings and disk models introduced a systematic error of about 2 % to 3 %.

The main problem consists in the fact that the formulas for the circles and rings models (Eq. 12) are incorrect and lead to an infinite limit when τ tends toward zero [10]. Taking into account the plots in Fig. 2, we can obviate the problem by introducing the modified shape function (Eq. 19). Until now, this was the only way to evaluate TPS measurements.

References

1. Y. Li, Ch. Shi, J. Liu, E. Liu, J. Shao, Z. Chen, D.J. Dorantes-Gonzalez, X. Hu, Meas. Sci. Technol. **25**, 015006 (2014)
2. Ľ. Kubičár, V. Boháč, in *Proceedings of 24th Internatinal Conference on Thermal Conductivity/12th Int. Thermal Expansion Symposium*, ed. by P.S. Gaal, D.E. Apostolescu (Technomic Pub., Lancaster, PA, 1999), p. 135
3. S.E. Gustafsson, Rev. Sci. Instrum. **62**, 797 (1991)
4. ISO, *Plastics—Determination of Thermal Conductivity and Thermal Diffusivity—Part 2*, ISO 22007-2 (ISO, Geneva, 2008)
5. H.S. Carslaw, J. Jaeger, *Conduction of Heat in Solids* (Clarendon, Oxford, 1959)
6. U. Hammerschmidt, W. Sabuga, Int. J. Thermophys. **21**, 217 (2000)
7. E. Karawacki, B.M. Suleiman, I. Ul-Haq, B. Nhi, Rev. Sci. Instrum. **63**, 4390 (1992)
8. M.K. Gustavsson, S.E. Gustafsson, in *Proceedings of 27th International Conference on Thermal Conductivity/15th Int. Thermal Expansion Symposium*, ed. by H. Wang, W. Porter (Knoxville, TN, 2003), p. 339
9. S. Malinarič, Int. J. Thermophys. **34**, 1953 (2013)
10. Y. Jannot, Z. Acem, Meas. Sci. Technol. **18**, 1229 (2007)