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# Rapid thermal conductivity measurement with a hot disk sensor
# Part 1. Theoretical considerations

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**Abstract**

The hot disk technique represents a transient plane source method for rapid thermal conductivity and thermal diffusivity measurement. The main advantages of the hot disk technique include: wide thermal conductivity range, from 0.005 W/(m K) to 500 W/(m K); wide range of materials types, from liquid, gel to solid; easy sample preparation; non-destructive; and more importantly, high accuracy. In this paper, the basic theory of thermal conductivity measurement with hot disk sensor will be discussed. Starting from the instantaneous point source solution, the mathematical expression of the average temperature change in the sensor surface during a hot disk measurement will be derived. This temperature change, which can be accurately determined by measuring the electrical resistance of the sensor, is highly dependent on the thermal transport properties of the surrounding material. By analyzing this temperature change as a function of time, it is possible to deduce the thermal conductivity and the thermal diffusivity of the surrounding material. Several practical considerations, from sample size requirement to the elimination of thermal contact resistance, will also be discussed.


*Keywords:* Thermal conductivity; Hot disk technique; Thermal diffusivity

## 1. Introduction

Thermal conductivity is a key thermal transport property of materials. Knowledge of a material's thermal conductivity is crucial for a wide range of applications, including polymer injection molding, home insulation using various building materials, insulation for space shuttle, and thermal management of electronic packages in semiconductor industry, etc.

There are two main categories of techniques to measure thermal conductivity, steady-state techniques and transient techniques. The radial heat flow method and the guarded hot-plate method are examples of steady-state techniques [1]. Hot wire and laser flash are examples of the transient technique [1]. The comparison of various transient techniques can be found elsewhere [2]. Other transient techniques include the 3ω method [3,4], the differential photoacoustic method [5], the pulsed photothermal displacement technique [6], and the thermal-wave technique [7]. Other techniques can be found in ref. [8].

Recently, the hot disk technique, which is a transient plane source technique, has gained popularity as a tool for rapid and accurate thermal conductivity measurement. A precursor to this technique was first developed by Gustafsson in 1967 as a non-steady-state method of measuring thermal conductivity of transparent liquids [9]. In the original experimental design, a thin rectangular metallic foil suspended in the liquid was heated by a constant electric current. The temperature distribution around the foil was measured optically as a function of time. From this result, both thermal diffusivity and thermal conductivity of the liquid could be determined.

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In principle, the hot strip sensor does not have to be rectangular and can have any form. An alternative is the hot disk sensor, which is made of a double spiral of thin nickel wire [10–12]. The thickness of the nickel wire is only 10 μm. The advantage of the hot disk sensor over the hot strip sensor will be discussed later. The hot disk technique, together with other types of transient plane source technique [13], have been successfully used to measure the thermal conductivity of a wide range of materials, including materials with low electrical conductivity (such as fused quartz) [14], building materials (e.g. cement and brick powder) [15,16], stainless steel [17], copper powder [18], anisotropic solids (crystalline quartz) [19], and thin metallic materials [20]. The hot disk technique represents a rapid and precise method for studying thermal transport properties of a wide range of materials. With suitable sample preparation, the hot disk technique can be used to measure thermal conductivity in the range of 0.005 W/(m K) to 500 W/(m K) over a wide temperature range. However, this technique has been largely kept as an academic research tool for many years, and was not commercially available until recently. Since its introduction to the commercial market, the hot disk technology was met with great enthusiasm [21].

The hot disk technique and other transient plane source techniques are based on using a thin metal strip or a metal disk as a continuous plane heat source. The metal disk or strip is sealed between two thin polyimide films for electrical insulation. During the experiment, the hot disk sensor is sandwiched between two pieces of samples to be investigated, and a small constant current is supplied to the sensor. The sensor also serves as a temperature monitor, so that the temperature increase in the sensor is accurately determined through resistance measurement. This temperature increase is highly dependent on the thermal transport properties of the material surrounding the sensor. By monitoring this temperature increase over a short period of time after the start of the experiment, it is possible to obtain precise information on the thermal transport properties of the surrounding material.

In this paper, the theoretical background of thermal conductivity measurement using the hot disk sensor will be discussed. In the theoretical treatment, a hot disk sensor can be approximated as a sensor with a number of concentric rings. Starting from the equation of heat conduction and its instantaneous point source solution, the mathematical expression of the temperature increase in the sensor surface can be obtained by integrating the point source solution over the source volume and time. This expression, which directly relates the temperature increase in the sensor surface to the sensor configuration, the output power, and the transport properties of the surrounding material, forms the basis of the hot disk measurement. Although the final result of the average temperature increase in a hot disk sensor has been given before [10], however, the detailed mathematical derivation was not presented in the literature, which will be discussed in this paper. As a special case, the mathematical treatment for measuring thin slab samples will also be discussed using the image source technique.

## 2. Theory

### *2.1. Heat conduction in an isotropic material: general equation and solution*

The differential equation of heat conduction in an isotropic material whose thermal conductivity is independent of temperature is given by [22]:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa}\frac{\partial T}{\partial t}, \tag{1}$$

where $T(x, y, z, t)$ is the temperature at point $(x, y, z)$ and time $t$, and

$$\kappa = \frac{K}{\rho c} \tag{2}$$

is the thermal diffusivity, $K$ the thermal conductivity, $\rho$ the density, and $c$ is the specific heat of the conducting material at temperature $T$. $\rho c$ is sometimes called the volumetric specific heat of the material and the units of $\rho c$ are J m$^{-3}$ K$^{-1}$. For a small change in temperature, we will assume that both $\rho$ and $c$ are temperature independent.

When a heat source of strength $Q$ is switched on at $t = 0$ in the material, Eq. (1) can be modified to include the effect of the heat source [22]:

$$\kappa \nabla^2 T + \frac{Q}{\rho c} = \frac{\partial T}{\partial t}. \tag{1a}$$

Usually $Q = Q(\vec{r}, t)$ is a function of position and time. $Q$ is the amount of heat released at $(x, y, z, t)$ per unit time per unit volume, or power dissipation per unit volume. The units of $Q$ are J s$^{-1}$ m$^{-3}$.

It is well known that the fundamental solution (for $Q = 0$) to Eq. (1) is given by

$$T = T_0 + \frac{1}{(4\pi\kappa t)^{3/2}} \exp\left(-\frac{r^2}{4\kappa t}\right), \quad (t > 0) \tag{3}$$

where $T_0$ is the initial temperature. In the case a source of strength $Q$ exists in the material, the general solution to Eq. (1a) is given by the convolution of the function $Q/\rho c$ with the fundamental solution expressed in Eq. (3):

$$T(\vec{r}, t) = T_0 + \int_0^t \int_{V'} \frac{Q(\vec{\xi}, t')}{\rho c} \frac{1}{[4\pi\kappa(t - t')]^{3/2}} \times \exp\left(-\frac{(\vec{r} - \vec{\xi})^2}{4\kappa(t - t')}\right) \mathrm{d}^3\vec{\xi}\, \mathrm{d}t'. \tag{4}$$

The volume integration is over the source volume $V'$. In the case that the source is an instantaneous point source at $\vec{r}_0 = (x_0, y_0, z_0)$ and is switched on only at $t' = 0$, then $Q(\vec{\xi}, t') = Q_0\delta(\vec{\xi} - \vec{r}_0)\delta(t')$, where $\delta(x)$ is the Dirac delta function. From Eq. (4), the instantaneous point source solution is then obtained as

$$T(\vec{r},t) = T_0 + \frac{Q_0/\rho c}{(4\pi\kappa t)^{3/2}} \exp\left(-\frac{(\vec{r}-\vec{r}_0)^2}{4\kappa t}\right). \tag{5}$$

It should be mentioned that because of the integrations over volume and time, the units of $Q_0/\rho c$ in Eq. (5) become K m$^3$. One can verify that in magnitude, $Q_0$ is the total amount of heat released by the point source:

$$H = \int_{V'} \int_0^t Q(\vec{\xi}, t')\, d^3\vec{\xi}\, dt' = Q_0. \tag{6}$$

Using Eq. (5), one can show that this heat indeed causes the temperature in the sample to increase by $\Delta T = T(\vec{r}, t) - T_0$, because

$$H = \int_{-\infty}^{\infty} \rho c \Delta T\, d^3\vec{r} = Q_0, \tag{7}$$

if one uses the fact that

$$\int_{-\infty}^{\infty} \exp\left(-\frac{(x-x_0)^2}{4\kappa t}\right) dx = (4\pi\kappa t)^{1/2}. \tag{8}$$

Thus, the general solution expressed in Eq. (4) is the integration of the instantaneous point source solution over time and the source volume.

### 2.2. Single ring source

A hot disk sensor composed of a double spiral nickel wire can be treated as a disk consisting of a certain number ($m$) of concentric rings [10]. Let us first consider an instantaneous single ring source with radius $a$ in the $z' = 0$ plane. If we use the cylindrical coordinates, the strength of the source can be expressed as

$$Q = Q_0 \delta(r'-a)\delta(z')\delta(t') . \tag{9}$$

The total heat released up to time $t$ by the instantaneous single ring source is then

$$H = \int_{V'} \int_0^t Q\, dV'\, dt' = \int_0^{\infty} Q r'\, dr' \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} dz' \int_0^t dt' = 2\pi a Q_0. \tag{10}$$

Thus, in magnitude, $Q_0$ is the heat released by per unit length of the ring source.

In the cylindrical coordinates, any position in the sample is $\vec{r} = (r, \theta, z)$, and any position in the source is $\vec{\xi} = (r', \theta', z')$, it is easy to verify that $(\vec{r}-\vec{\xi})^2 = r^2 + r'^2 - 2rr' \cos(\theta - \theta') + (z-z')^2$. If we assume the source is in the $z' = 0$ plane, using Eq. (4), the temperature increase caused by this instantaneous ring source becomes [22]:

$$\begin{aligned} T(r,\theta,z,t) - T_0 &= \frac{1}{\rho c (4\pi\kappa t)^{3/2}} \int_0^{\infty} e^{-[r^2+r'^2-2rr' \cos(\theta-\theta')]/4\kappa t} \\ &\quad \times Q_0 \delta(r'-a)\delta(z') r'\, dr' \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} e^{-(z-z')^2/4\kappa t}\, dz' \\ &= \frac{e^{-z^2/4\kappa t} e^{-(r^2+a^2)/4\kappa t} Q_0}{\rho c (4\pi\kappa t)^{3/2}} \int_0^{2\pi} e^{ra \cos(\theta-\theta')/2\kappa t} a\, d\theta' \\ &= \frac{2\pi a Q_0\, e^{-(r^2+a^2+z^2)/4\kappa t}}{\rho c (4\pi\kappa t)^{3/2}} I_0\left(\frac{ra}{2\kappa t}\right) \end{aligned} \tag{11}$$

where

$$I_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{x \cos\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{x \sin\theta}\, d\theta \tag{12}$$

is the first kind modified Bessel function of the zeroth order.

When the single ring source is continuous, the source strength can be expressed as

$$Q = Q_0 \delta(r'-a)\delta(z') u(t'), \tag{13}$$

where $u(t')$ is the Heaviside unit step function:

$$u(t') = \begin{cases} 0 & \text{for } t' < 0 \\ 1 & \text{for } t' \geq 0 \end{cases} . \tag{14}$$

The source is turned on at $t' = 0$ and stays on afterwards. It can be verified that the total amount of heat released by the source up to time $t$ is

$$\begin{aligned} H(t) &= \int_{V'} \int_0^t Q\, dV'\, dt' \\ &= \int_0^{\infty} Q_0 \delta(r'-a) r'\, dr' \int_0^{2\pi} d\theta' \int_{-\infty}^{\infty} \delta(z')\, dz' \\ &\quad \times \int_0^t u(t')\, dt' = 2\pi a Q_0 t. \end{aligned} \tag{15}$$

Therefore, it becomes clear that $2\pi a Q_0$ is the magnitude of power dissipation associated with the source, or $P_o = 2\pi a Q_0$.

To obtain the temperature increase caused by a continuous ring source, one only needs to change $t$ to $t - t'$ in Eq. (11) and integrate over time $t'$ to obtain the temperature solution:

$$\begin{aligned} T(r,\theta,z,t) - T_0 &= \frac{2\pi a Q_0}{\rho c (4\pi\kappa)^{3/2}} \int_0^t \frac{e^{-(r^2+a^2+z^2)/4\kappa(t-t')}}{(t-t')^{3/2}} \\ &\quad \times I_0\left(\frac{ra}{2\kappa(t-t')}\right) dt'. \end{aligned} \tag{16}$$

In Eq. (11), $2\pi a Q_0$ is the total amount of heat released by the source and it has the unit of Joule, but in Eq. (16), $2\pi a Q_0$ is the magnitude of the power output of the source, which has units of J s$^{-1}$, because the integration over time has yet to be carried out.

### 2.3. Hot disk sensor

A hot disk sensor with a double spiral of nickel wire can be treated as a sensor with $m$ concentric rings which are equally spaced, since the sensor is designed to have uniform power density throughout the disk. Assume that $a$ is the radius of the largest ring, then the smallest ring has a radius of $a/m$. The total length of the heating filament is

$$L = \sum_{l=1}^{m} 2\pi l \frac{a}{m} = (m+1)\pi a. \tag{17}$$

Suppose the source is continuous and it is turned on at $t=0$. It can then be represented as

$$Q = Q_0 \sum_{l=1}^{m} \delta\left(r' - \frac{la}{m}\right)\delta(z')u(t'). \tag{18}$$

Similarly, one can verify that the total heat released by the sensor up to time $t$ is

$$\begin{aligned} H &= \int_{V'}\int_0^t Q(\vec{\xi}, t')\,\mathrm{d}V'\,\mathrm{d}t \\ &= \int_0^{\infty} Q_0 \sum_{l=1}^{m}\delta\left(r' - \frac{la}{m}\right)\delta(z')r'\,\mathrm{d}r' \int_0^{2\pi}\mathrm{d}\theta' \\ &\quad \times \int_{-\infty}^{\infty}\mathrm{d}z' \int_0^t u(t')\,\mathrm{d}t' \\ &= \pi a(m+1)Q_0 t = LQ_0 t, \end{aligned} \tag{19}$$

just as expected. Again, $Q_0$ is the heat released per unit length per unit time of the sensor coil, and $\pi a(m+1)Q_0 = P_\mathrm{o}$ is the power output of the hot disk sensor.

The temperature increase caused by the hot disk sensor can be simply obtained by carrying out the integration in Eq. (4), noting that the source strength is now expressed by Eq. (18). This can be done by substituting $a$ by $la/m$ and carry out the summation over $l$ in Eq. (16). The result is

$$\begin{aligned} \Delta T(r, z, t) &= \frac{2\pi Q_0}{\rho c}\int_0^t \frac{\mathrm{d}t'}{[4\pi\kappa(t-t')]^{3/2}} \\ &\quad \times \sum_{l=1}^{m}\frac{la}{m}\,\mathrm{e}^{-(r^2+(l^2a^2/m^2)+z^2)/4\kappa(t-t')} \\ &\quad \times I_0\left(\frac{rla}{2m\kappa(t-t')}\right) \\ &= \frac{P_\mathrm{o}}{\rho c m(m+1)}\int_0^t \frac{\mathrm{d}t'}{4[\pi\kappa(t-t')]^{3/2}} \\ &\quad \times \sum_{l=1}^{m} l\,\mathrm{e}^{-(r^2+(l^2a^2/m^2)+z^2)/4\kappa(t-t')} \\ &\quad \times I_0\left(\frac{rla}{2m\kappa(t-t')}\right), \end{aligned} \tag{20}$$

where we have used the fact that $P_\mathrm{o} = \pi a(m+1)Q_0$, as we discussed in Eq. (19).

In a hot disk measurement, we are only concerned about the temperature change near the surface of the sensor. Thus, we can let $z \to 0$ and Eq. (20) becomes

$$\begin{aligned} \Delta T(r, t) &= \frac{P_\mathrm{o}}{\rho c m(m+1)}\int_0^t \frac{\mathrm{d}t'}{4[\pi\kappa(t-t')]^{3/2}} \\ &\quad \times \sum_{l=1}^{m} l\,\mathrm{e}^{-(r^2+(l^2a^2/m^2))/4\kappa(t-t')} I_0\left(\frac{rla}{2m\kappa(t-t')}\right). \end{aligned} \tag{21}$$

Let us introduce a new integration variable $\sigma$, and let

$$\sigma^2 = \frac{\kappa(t-t')}{a^2} \tag{22}$$

we have $\mathrm{d}t' = -2\sigma a^2\,\mathrm{d}\sigma/\kappa$, when $t'=0$, $\sigma = \sqrt{\kappa t}/a$; when $t'=t$, $\sigma=0$. Therefore, Eq. (21) becomes

$$\begin{aligned} \Delta T(r, t) &= \frac{P_\mathrm{o}}{2\pi^{3/2} m(m+1)\rho c}\int_{\sqrt{\kappa t}/a}^{0}\left(\frac{-\mathrm{d}\sigma}{\kappa\sigma^2 a}\right) \\ &\quad \times \sum_{l=1}^{m} l\,\mathrm{e}^{-((r^2/a^2)+(l^2/m^2))/4\sigma^2} I_0\left(\frac{rl}{2ma\sigma^2}\right) \\ &= \frac{P_\mathrm{o}}{2\pi^{3/2} a m(m+1)K}\int_0^{\tau}\frac{\mathrm{d}\sigma}{\sigma^2} \\ &\quad \times \sum_{l=1}^{m} l\,\mathrm{e}^{-((r^2/a^2)+(l^2/m^2))/4\sigma^2} I_0\left(\frac{rl}{2ma\sigma^2}\right), \end{aligned} \tag{23}$$

where

$$\tau = \frac{\sqrt{\kappa t}}{a} \tag{24}$$

is a dimensionless parameter called the characteristic time ratio, and $K = \kappa\rho c$ is the thermal conductivity of the material.

Eq. (23) describes the temperature increase at any point in the $z=0$ plane (i.e. sensor surface) after the current to the hot disk sensor is turned on. However, during a hot disk measurement, we can only measure the temperature increase for the sensor itself. Thus, we need to determine the average temperature increase for the sensor only. This can be done by averaging $\Delta T(r, \tau)$ over the length of the concentric rings:

$$\Delta\bar{T}(\tau) = \frac{1}{L}\int_0^{2\pi}\Delta T(r, \tau)\sum_{k=1}^{m}\delta\left(r - \frac{k}{m}a\right) r\,\mathrm{d}\theta \tag{25}$$

Substituting Eq. (23) into Eq. (25) and using the fact that $L=(m+1)\pi a$, we can express the average temperature increase in the sensor surface as

$$\begin{aligned} \Delta\bar{T}(\tau) &= \frac{1}{(m+1)\pi a}\frac{P_\mathrm{o}}{2\pi^{3/2} a m(m+1)K}\int_0^{\tau}\frac{\mathrm{d}\sigma}{\sigma^2}\sum_{k=1}^{m}\frac{ka}{m} \\ &\quad \times \sum_{k=1}^{m} l\,\mathrm{e}^{-((k^2/m^2)+(l^2/m^2))/4\sigma^2} I_0\left(\frac{kl}{2m^2\sigma^2}\right) 2\pi \\ &= \frac{P_\mathrm{o}}{\pi^{3/2} a K} D(\tau), \end{aligned} \tag{26}$$

where $D(\tau)$ is a dimensionless time function given by

$$D(\tau) = \frac{1}{m^2(m+1)^2} \int_0^{\tau} \frac{d\sigma}{\sigma^2} \sum_{k=1}^{m} k \sum_{l=1}^{m} l\, e^{-((k^2+l^2)/m^2)/4\sigma^2} \times I_0\left(\frac{kl}{2m^2\sigma^2}\right), \quad (27)$$

which is exactly the expression given in refs. [10,11,23].

From Eq. (26), we can see that the average temperature increase in the hot disk sensor is proportional to a function $D(\tau)$, which is a rather complicated function of a dimensionless parameter $\tau = \sqrt{\kappa t}/a$, but, numerically, it can be accurately evaluated to five or six significant figures.

When using the hot disk technique to determine thermal transport properties, a constant electric current is supplied to the sensor at time $t = 0$, then the temperature change of the sensor is recorded as a function of time. The average temperature increase across the hot disk sensor area can be measured by monitoring the total resistance of the hot disk sensor:

$$R = R_0[1 + \alpha\Delta\overline{T}(t)], \quad (28)$$

where $R$ is the total electrical resistance at time $t$, $R_0$ is the initial resistance at $t = 0$, $\alpha$ is the temperature coefficient of resistivity, which is well known for nickel. Eq. (28) allows us to accurately determine $\Delta\overline{T}$ as a function of time.

If one knows the relationship between $t$ and $\tau$, one can plot $\Delta\overline{T}$ as a function of $D(\tau)$, and a straight line should be obtained. The slope of that line is $P_0/(\pi^{3/2}aK)$, from which thermal conductivity $K$ can be calculated. However, the proper value of $\tau$ is generally unknown, since $\tau = \sqrt{\kappa t}/a$ and the thermal diffusivity $\kappa$ is unknown. To calculate the thermal conductivity correctly, one normally makes a series of computational plots of $\Delta\overline{T}$ versus $D(\tau)$ for a range of $\kappa$ values. The correct value of $\kappa$ will yield a straight line for the $\Delta\overline{T}$ versus $D(\tau)$ plot. This optimization process can be done by the software until an optimized value of $\kappa$ is found. In practice, we can measure the density and the specific heat of the material separately, so that between $K$ and $\kappa$, there is only one independent parameter. Therefore, both thermal conductivity and thermal diffusivity of the sample can be obtained from above procedure based on the transient measurement using a hot disk sensor.

## 3. Thin slab samples: a special case

Above analysis is based on the assumption that from the hot disk sensor point of view, the sample dimensions are infinite, or they can be practically considered infinite within the short measurement time, so that the sample boundaries do not affect the temperature increase measured by the hot disk sensor. The sample size requirement for a standard hot disk measurement will be discussed later. For a thin slab sample or a highly conductive material, it is impractical to shorten the measurement time so that during this period, the sample seems infinite. In this case, the hot disk technique can still be applied to measure the thermal transport properties of thin slab materials. In the case that the effect of sample thickness cannot be ignored, the following procedure will be used. Again, the sensor is sandwiched between two identical thin slab samples with thickness $h$. The outer surfaces of the two slabs are covered with isolation material so that no heat can transfer out of these surfaces during measurement, as illustrated in Fig. 1. Theoretically, this condition can be satisfied by introducing the image sources of the same strength as the original real source ($Q_0 = P_o/((m+1)\pi a)$) at $z' = \pm 2h$, $\pm 4h$, ... planes, assuming the sample is infinitely thick, as shown in Fig. 1. Similar to Eq. (18), the sources can be expressed as

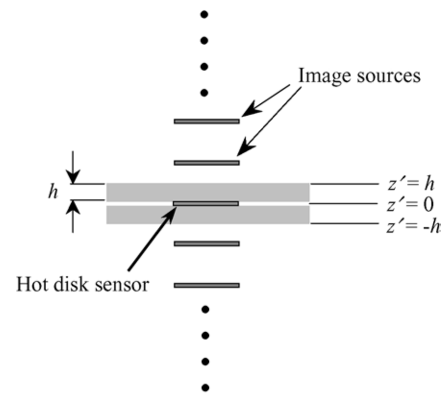


Fig. 1. To measure thermal conductivity of a thin slab sample, the hot disk sensor is sandwiched between two specimens with thickness $h$. The outer surfaces of the specimens, $z' = \pm h$, are insulated. Theoretically, this condition can be achieved by introducing infinite number of image sources at $z' = \pm 2h$, $z' = \pm 4h$, etc., in an infinitely thick sample, as illustrated above.

$$Q = Q_0 \sum_{n=-\infty}^{\infty} \sum_{l=1}^{m} \delta\left(r' - \frac{la}{m}\right) \delta(z' - 2nh) u(t'). \quad (29)$$

Again, $u(t')$ is the Heaviside unit step function. The term $n = 0$ in Eq. (29) represents the original real source, the rest are all image sources. Substituting this expression into Eq. (4) and using the result obtained in Eq. (20), we have

$$\Delta T(r, z, t) = \frac{2\pi a Q_0}{\rho c m (4\pi\kappa)^{3/2}} \int_0^t \frac{dt'}{(t-t')^{3/2}} \times \sum_{n=-\infty}^{\infty} e^{-(z-2nh)^2/4\kappa(t-t')} \times \sum_{l=1}^{m} l\, e^{-(r^2+(l^2a^2/m^2))/4\kappa(t-t')} I_0\left(\frac{rla}{2\kappa(t-t')m}\right) \quad (30)$$

Eq. (30) gives the temperature change at time $t > 0$ at any point $(r, \theta, z)$ in the slab sample caused by a hot disk sensor source at $z' = 0$ plane, with the condition that the outer surfaces of the two slabs are insulated. Again, above equation can be modified if we are only interested in the temperature change near the sensor surface, i.e., at $z = 0$ plane, and notice that the power output of the sensor is $P_o = \pi a(m+1)Q_0$:

$$\Delta T(r,t) = \frac{1}{4(\pi\kappa)^{3/2}} \frac{P_o}{m(m+1)\rho c} \times \int_0^t \frac{dt'}{(t-t')^{3/2}} \left(1 + 2\sum_{n=1}^{\infty} e^{-n^2h^2/\kappa(t-t')}\right) \times \sum_{l=1}^{m} l\, e^{-(r^2+(l^2a^2/m^2))/4\kappa(t-t')} I_0\left(\frac{rla}{2m\kappa(t-t')}\right). \quad (31)$$

In Eq. (31) we used the fact that

$$\lim_{z\to 0} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{(z-2nh)^2}{4\kappa(t-t')}\right) = 1 + 2\sum_{n=1}^{\infty} \exp\left(-\frac{n^2h^2}{\kappa(t-t')}\right). \quad (32)$$

Similar to previous procedure, we can introduce $\sigma^2 = \kappa(t-t')/a^2$ and define $\tau = \sqrt{\kappa t}/a$, and finally, the temperature change due to a continuous hot disk sensor sandwiched between two slab samples can be obtained as

$$\Delta T(r,\tau) = \frac{P_o}{2\pi^{3/2}am(m+1)K} \times \int_0^\tau \frac{d\sigma}{\sigma^2} \left(1 + 2\sum_{n=1}^{\infty} e^{-(n/\sigma)^2(h/a)^2}\right) \times \sum_{l=1}^{m} l\, e^{-((r^2/a^2)+(l^2/m^2))/4\sigma^2} I_0\left(\frac{rl}{2ma\sigma^2}\right), \quad (33)$$

Again, we proceed to average the temperature rise over the $m$ concentric rings and finally obtained

$$\Delta\bar{T}(\tau) = \frac{P_o}{\pi^{3/2}aK} D_s(\tau), \quad (34)$$

where

$$D_s(\tau) = \frac{1}{m^2(m+1)^2} \int_0^\tau \frac{d\sigma}{\sigma^2} \left(1 + 2\sum_{n=1}^{\infty} e^{-(n/\sigma)^2(h/a)^2}\right) \times \sum_{k=1}^{m} k \sum_{l=1}^{m} l\, e^{-((k^2+l^2)/m^2)/4\sigma^2} I_0\left(\frac{kl}{2m^2\sigma^2}\right) \quad (35)$$

Eq. (35) is exactly the expression given in ref. [20]. As we can see from the above equation, the summation over $n$ ($n \neq 0$) represents the contribution of all image sources. For large $h$, these contributions become negligible, and above equation becomes the same as Eq. (27), which is the case for a thick sample. As $n$ increases, $e^{-(n/\sigma)^2(h/a)^2}$ decreases quickly. Thus, in practice, it is only necessary to consider the first few terms when evaluating the contribution of the image sources.

Based on Eq. (34), similar optimization procedure can be performed until a proper thermal diffusivity $\kappa$ is found so that a straight line is obtained for the $\Delta\bar{T}$ versus $D_s(\tau)$ plot. From the slope of this line, thermal conductivity value can be obtained for the thin slab samples.

## 4. Probing depth

The theoretical analysis discussed above assumes that the sensor is placed in a sample that is infinitely large. For thin slab samples, it is assumed that the sample dimensions in the $xy$-plane are infinite. This is not the case in reality since all sample sizes are limited. Therefore, it is essential that during the measurement time $t$, the sample boundaries have little influence on the measurement result. One can define a probing depth $\Delta p$, which is the distance from the sensor edge to the nearest free surface of the sample. Analysis has shown that if [17,24]:

$$\Delta p \geq \sqrt{4\kappa t} \quad (36)$$

then the influence of the sample size on the final result will be negligible. This criterion will be used as a guideline to prepare samples with dimensions suitable for hot disk measurement. For thin slab samples, $\Delta p$ defines the required minimum distance between the edge of the sensor and the nearest sample boundary in the $xy$-plane.

## 5. Time correction

In the ideal situation, the temperature response of the sample is assumed to be instantaneous when the current to the hot disk sensor is switched on. In reality, however, there are a number of factors which would affect the temperature response time, including non-ideal electric components; the heat capacity of the sensor and the insulation material (Kapton™ or polyimide); time delay caused by thermal resistance between the sensor and the material; and intrinsic instrument dead time [17]. For these reasons, a time correction, $t_c$, is needed when evaluating the thermal conductivity using Eq. (26):

$$\Delta\bar{T}(\tau_c) = \frac{P_o}{\pi^{3/2}aK} D(\tau_c), \quad (37)$$

where $\tau_c = \sqrt{\kappa(t-t_c)}/a$ is the corrected characteristic time ratio. The time correction can be obtained by least square fitting so that the average temperature increase is linearly dependent on the function $D(\tau_c)$ [17]. Based on our experimental results, the typical time correction is 50–100 ms.

## 6. Contact resistance

Contact resistance is the interfacial thermal resistance between the sample surface and the sensor. It is present in almost all measurement techniques, except the non-contact techniques (such as laser flash). In the hot disk technique, the presence of insulating layers (Kapton™ or polyimide) adds an additional contact resistance between the material and the sensing disk. However, it was noted that the influence of contact resistance on the average temperature increase becomes a constant, $\Delta T$, after a short period of time [17,25]:

$$\Delta\bar{T}(\tau_c) = \Delta T + \frac{P_o}{\pi^{3/2} a K} D(\tau_c), \tag{38}$$

where $\Delta T$ is inversely proportional to the thermal conductivity of the insulating layer, and is related to the dimensions of the insulating layer. This time can be estimated as $\Delta t_i = \delta^2/\kappa_i$, where $\delta$ is the thickness of the insulation layer and $\kappa_i$ is the thermal diffusivity of the insulation material [26]. Numerical simulations have shown that for a 25 μm thick Kapton™ insulation layer, the time needed for $\Delta T$ to become a constant is typically ∼50 ms [21,27]. Thus, this term can be easily separated in the software when $\Delta\bar{T}(\tau_c)$ versus $D(\tau_c)$ is plotted using data points generated after $t > 50$ ms (and after time correction, which is also small).

## 7. Temperature drift

Before an actual measurement, any systematic temperature drift surrounding the hot disk sensor is monitored for 25 s. Any systematic temperature drift, if detected, will be corrected when thermal conductivity and thermal diffusivity are calculated.

## 8. Heat capacity of the sensor

Because the hot disk sensor has its own heat capacity that can influence the temperature change of the sensor, heat capacity of the sensor itself (including nickel wire and the polyimide insulation layer) is corrected. The calibrated heat capacity of the sensor is included as a default value in the analysis software.

## 9. Density and heat capacity of the sample

In principle, it is not necessary to know a material's specific heat and density in order to calculate thermal conductivity when bulk samples are available for the measurement. After the raw experimental data are collected, a linear regression is performed to obtain the optimized $\tau$ value so that a linear relationship is obtained for $\Delta\bar{T}(\tau_c)$ versus $D(\tau_c)$. Thermal diffusivity, $\kappa$, of the material can be easily calculated when $\tau$ is known. Then, thermal conductivity $K$ of the material can be obtained from the slope of the linear fit. When both $\kappa$ and $K$ are known, the volumetric specific heat, $\rho c$, can be determined. It is, however, always a good idea if one can determine specific heat and density separately using other experimental techniques. When $\rho$ and $c$ are known, there is only one independent fitting parameter. In practice, when bulk samples are used, the thermal conductivity value can be determined with or without the prior knowledge of $\rho c$, and the difference between these two approaches is small.

## 10. Hot disk versus hot strip

In the original transient plane source technique, a rectangular metal foil was used as the sensor [9,14,15,17]. This was also called the hot strip technique. Mathematical analysis for hot strip sensor is simpler than that for hot disk sensor. For hot strip sensor, the voltage or resistance of the sensor can be expressed as an analytical function of the characteristic time $\tau$ based on first order approximation. For small values of $\tau$, this expression can further be simplified, even for second order approximation [28]. For the hot disk sensor, no analytical expression has been obtained for the function $D(\tau)$, even for small values of $\tau$ [29].

However, there are two main advantages for using the hot disk sensor. The first one is that because of the design, the hot disk sensor has a much higher resistance than that of the hot strip sensor. Therefore, the temperature measurement, which is done by measuring the sensor resistance, can be performed with higher sensitivity and accuracy. The second advantage in using the hot disk sensor is that a much more compact sample can be studied without violating the sample size requirement [29]. This is because in the hot strip method, theoretical analysis assumes that the strip is infinitely long. In practice, it is often required that the length to width ratio is 20:1. Under this condition, the corresponding sample size will be much larger than the size required for the hot disk measurement.

## 11. Conclusions

Theory of the hot disk technique for thermal conductivity measurement was discussed from the first principles. The hot disk sensor, which is composed of a double spiral nickel wire, is approximated as $m$ concentric rings. The sensor is used as a heat source as well as a temperature monitor. During the thermal conductivity measurement, the sensor is sandwiched between two halves of the sample. A constant current is supplied to the sensor. During the measurement, the sensor temperature as a function of time $t$ or characteristic time $\tau$ has a strong dependence on the thermal transport properties of the surrounding material. Starting from the point source solution for thermal conduction is an infinite isotropic substance, the average temperature increase $\Delta\bar{T}$ near the sensor surface is obtained as a function of $\tau$. The result showed that $\Delta\bar{T}$ is proportional to a complicated function $D(\tau)$, which can be calculated numerically if the sensor configuration is known. When proper $\tau$ value is chosen based on optimization, as described before, the slope of $\Delta\bar{T}$ versus $D(\tau)$ is inversely proportional to thermal conductivity of the sample.

As a special case, the thermal conductivity measurement of thin slab samples can be performed by insulating the outer surfaces of the sample. Theoretically, the conduction problem can be solved by introducing the image sources, and a modified function $D_s(\tau)$ can be obtained. Again, the slope of $\Delta\bar{T}$ versus $D_s(\tau)$ is inversely proportional to thermal conductivity of the thin slab sample.

With proper corrections, the hot disk technique provides an excellent tool for rapid and accurate measurement for both thermal conductivity and thermal diffusivity of a wide range of materials. This technique is capable of measuring the thermal conductivity over a wide range with high accuracy, and the typical measurement time is 2.5–5 s. The hot disk technique is a valuable tool for material inspection and selection.

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