Exact expressions for thermal contrast detected with thermal and quantum detectors

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ABSTRACT

The detected thermal contrast is a recently defined figure of merit introduced to describe the overall performance of a detector detecting radiation from a thermal source. We examine the detected thermal contrast for the case where the target emissivity can be assumed to be a function of the temperature and independent of the wavelength within a narrow wavelength interval of interest. Exact expressions are developed to evaluate the thermal contrast detected by both thermal and quantum detectors for focal-plane radiation detecting instruments. Expressions for the thermal contrast of a blackbody, an intrinsic radiative quantity of a body independent of the detection process, and simplified expressions for the detected thermal contrast for target emissivities which are well approximated by the grey body approximation are also given. It is found the contribution in the detected thermal contrast consists of two terms. The first results from changes occurring in the emissivity of a target with temperature while the second results from purely radiative processes. The size of the detected thermal contrast is found to be similar for the two detector types within typical infrared wavelength intervals of interest, contradicting a result previously reported in the literature. The exact results are presented in terms of a polylogarithmic formulation of the problem and extend a number of approximation schemes that have been proposed and developed in the past.

Keywords: thermal contrast, blackbody radiation, detectors, emissivity, radiometry, polylogarithms

1. INTRODUCTION

Thermal contrast is an important concept in infrared imaging as it allows the distinction between an object and its background to be made solely on the basis of its temperature. For a blackbody source the spectral thermal contrast is given by a change in the spectral radiant exitance resulting from a change in its temperature. In practical applications this idea is extended to a fractional thermal contrast within a given wavelength interval of interest. When it comes to real objects radiating thermally, radiation that originates at the surface of an object, is collected by a detector, and leads to a final, measurable output signal in some electro-optical system can be quantified by the *detected thermal contrast*. The detected thermal contrast is a recently introduced figure of merit[1] that attempts to describe the overall performance of the sequence of events from the initial emission of thermal radiation at the surface of a body (target) of interest to the final measurable output signal seen in the detecting instrument.*

Broadly speaking, detectors in the infrared can be divided into one of two common types depending on how the radiation is detected. The first are the so-called *thermal* detectors.[5] Here a heating effect caused by the incident radiation results in a variation in some physical parameter (usually electrical) of the detector with temperature. The second are the so-called *quantum* detectors.[6] Here there is a direct interaction between the incident photons and the electrons of the detector. In the former case the detector response is proportional to the energy absorbed while in the latter case it is proportional to the number of photons absorbed. Connecting particular units in a natural way to the type of detector used turns out to be useful when discussing the detected thermal contrast of a detector. *Radiometric* quantities, that is those

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*It should be pointed out another figure of merit that has been employed in the past to describe a detector's ability to detect thermal radiation is the so-called *thermal figure of merit*.[2–4] Denoted by M^*, while similar to the detected thermal contrast to be considered here, the two are not the same.

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quantities used to describe radiant energy, are based on the joule (or watt) and is the natural unit of choice for thermal detectors (so-called "energetic" units). *Actinometric* quantities, that is those quantities related to the measure of photons, are more appropriate for quantum detectors (so-called "photonic" units).[7]

We present expressions for the detected thermal contrast for the two important classes of detector type. Naturally, different types of detectors leads to different properties in the detected thermal contrast as the response with wavelength for each will be different. Expressions found for the detected thermal contrast for either detector type are however qualitatively similar in form. By including an emissivity for the body from where the thermal radiation originates from which is depend on temperature, in the limit of narrow wavelength intervals where the emissivity can be assumed to be spectrally constant, explicit, closed form expressions for the detected thermal contrast in terms of the polylogarithm function for both detector types can be found. In line with what has been shown before, the detected thermal contrast for either detector type consists of two terms.[8–10] The first is an emissivity dependent term brought about by changes in the temperature at the surface of the target while the second is a purely radiative term related to the thermal contrast for the corresponding blackbody. Greatly simplified expressions for the detected thermal contrast in the grey body approximation are also given. The analysis presented here extends several previous works given in the past which have made use of a number of different approximation schemes.[8–20] The present work also impugns several results previously reported in the literature relating to the behaviour of the detected thermal contrast as a function of the wavelength band selected and type of detector considered.

As an example of the work presented, thermal infrared systems used for observing common terrestrial scenes operating at room temperature are given. The expressions developed for the detected thermal contrast are however not limited to this regime but apply equally to infrared measurements of sources made over any spectral range and temperature of interest. Our work therefore applies with equal applicability to common high-temperature industrial and space applications as it does to familiar room-temperature terrestrial scenes.

2. THERMAL CONTRAST FOR A BLACKBODY

We begin by first considering the case of a radiating blackbody undergoing a change in temperature. We use the term *thermal contrast* to denote the change in the exitance of a blackbody arising from a change in its temperature. It is an intrinsic property for a blackbody and stands quite apart from the detection of radiation by a detector to be considered later. As we are going to be concerned with two different types of devices in the detection of radiation, two separate cases will be considered depending on the physical mechanism involved in the detection process; those of the thermal type and those of the quantum type. The thermal contrast of a blackbody for these two separate cases is now considered.

2.1 Radiometric case

For the radiometric case the spectral thermal contrast $\mathfrak{T}^{\rm b}_{{\rm e},\lambda}$ is given by the fractional change in spectral radiant exitance of a blackbody over the fractional change in the temperature of the blackbody, namely[21]

$$\mathfrak{T}^{\rm b}_{{\rm e},\lambda} = \frac{dM^{\rm b}_{{\rm e},\lambda}}{M^{\rm b}_{{\rm e},\lambda}} \Bigg/ \frac{dT}{T} = \frac{T\left(\dfrac{dM^{\rm b}_{{\rm e},\lambda}}{dT}\right)}{M^{\rm b}_{{\rm e},\lambda}}. \tag{1}$$

Here T is the absolute temperature of the body while $M^{\rm b}_{{\rm e},\lambda}$ is the spectral radiant exitance for a blackbody and is given by Planck's well-known law for thermal radiation

$$M^{\rm b}_{{\rm e},\lambda}(\lambda, T) = \frac{c_1}{\lambda^5\left[\exp\left(\dfrac{c_2}{\lambda T}\right) - 1\right]}. \tag{2}$$

The constants appearing in the above equation are the first ($c_1 = 3.741\,771\,53 \times 10^{-16}$ W m^2) and second ($c_2 = 1.438\,7770 \times 10^{-2}$ m K) radiation constants respectively.† As is customary, the subscript λ for wavelength indicates a

†All values quoted for the fundamental constants are the 2010 adjusted values recommended by the Committee on Data for Science and Technology (CODATA) for international use.[22]

spectral quantity is being considered, the subscript "e" indicates the radiometric case has been selected while the superscript "b" is used on account of the body under consideration is that of a blackbody. As a fractional ratio, the spectral thermal contrast is dimensionless.

Evaluating Eq. (1) for a blackbody one finds

$$\mathfrak{T}^{\rm b}_{{\rm e},\lambda} = \frac{c_2}{\lambda T} \frac{1}{1 - \exp\left(-\dfrac{c_2}{\lambda T}\right)}. \tag{3}$$

From Eq. (3) it is clear the spectral thermal contrast increases with a decrease in either the wavelength or the temperature. As we plan to use a polylogarithmic formulation throughout, let us rewrite Eq. (3) in terms of polylogarithm functions (see Appendix A for details on the polylogarithmic functions, $\text{Li}_s(x)$). Doing so yields

$$\mathfrak{T}^{\rm b}_{{\rm e},\lambda} = \frac{\zeta \, \text{Li}_{-1}({\rm e}^{-\zeta})}{\text{Li}_0({\rm e}^{-\zeta})}. \tag{4}$$

Here we have introduced the dimensionless parameter $\zeta = c_2/(\lambda T)$; a parameter to be used throughout the remainder of this work.

As with any spectral quantity, it is differential in nature, while real physical meaning can only be attached to the corresponding quantity specified within a finite spectral band.[23] For the case of thermal contrast, when the spectral interval $[\lambda_1, \lambda_2]$ (here $\lambda_2 > \lambda_1$) is considered one needs to replace the spectral radiant exitance $M^{\rm b}_{{\rm e},\lambda}$ in Eq. (1) with the radiant exitance within a given spectral interval, namely $M^{\rm b}_{{\rm e},\lambda_1 \to \lambda_2}$, where

$$M^{\rm b}_{{\rm e},\lambda_1 \to \lambda_2} = \int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda, T) \, d\lambda \, . \tag{5}$$

The thermal contrast within a finite wavelength interval $[\lambda_1, \lambda_2]$ can therefore be defined as

$$\mathfrak{T}^{\rm b}_{{\rm e},\lambda_1 \to \lambda_2} = \frac{T \dfrac{d}{dT} \displaystyle\int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda, T) \, d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda, T) \, d\lambda}. \tag{6}$$

The integral appearing in the denominator of Eq. (6) is well known as it often appears in problems related to the study of radiative heat transfer.[24] To help aid in the analysis which is to follow we introduce an important class of functions known as the *blackbody fractional functions*. For the radiometric case the fraction of the total radiant exitance emitted by a blackbody into the finite spectral band $[\lambda_1, \lambda_2]$ is given by the ratio

$$\mathfrak{F}_{{\rm e},\lambda_1 \to \lambda_2} = \frac{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda, T) d\lambda}{\displaystyle\int_0^{\infty} M^{\rm b}_{{\rm e},\lambda}(\lambda, T) d\lambda}. \tag{7}$$

It is a dimensionless quantity between zero and one and gives a measure of the fraction of energy radiated by a blackbody into a finite wavelength interval compared to the total amount radiated over all wavelengths. In following what has been done in the past, this quantity is referred to as the (blackbody) *fractional function of the first kind* corresponding to the radiometric case.[25,26] The calculation of this integral has a long and distinguished history[27] with a range of approximations being purposed for it in the past.[28–34] By application of the polylogarithm function it is now possible to express Eq. (7) in closed form. In doing so, by expressing the fractional function in terms of a single variable λT,[24] in conjunction with properties for the definite integral, one can write

$$\mathfrak{F}_{{\rm e},\lambda_1 \to \lambda_2} = \mathfrak{F}_{{\rm e},\lambda_1 T \to \lambda_2 T} = \mathfrak{F}_{{\rm e},0 \to \lambda_2 T} - \mathfrak{F}_{{\rm e},0 \to \lambda_1 T}, \tag{8}$$

where[35–37]

$$\mathfrak{F}_{e,0\to\lambda T} = \frac{15}{\pi^4}\left[\zeta^3\mathrm{Li}_1(\mathrm{e}^{-\zeta}) + 3\zeta^2\mathrm{Li}_2(\mathrm{e}^{-\zeta}) + 6\zeta\mathrm{Li}_3(\mathrm{e}^{-\zeta}) + 6\mathrm{Li}_4(\mathrm{e}^{-\zeta})\right]. \tag{9}$$

For the numerator of Eq. (6), applying Leibniz's rule for differentiating integrals followed by integration by parts reduces it to a form involving the fractional function of the first kind. The final result for the thermal contrast in closed form in terms of polylogarithms can therefore be written as

$$\mathfrak{T}^{\mathrm{b}}_{e,\lambda_1 T\to\lambda_2 T} = 4 + \frac{15}{\pi^4}\frac{\zeta_2^4\,\mathrm{Li}_0(\mathrm{e}^{-\zeta_2}) - \zeta_1^4\,\mathrm{Li}_0(\mathrm{e}^{-\zeta_1})}{\mathfrak{F}_{e,\lambda_1 T\to\lambda_2 T}}, \tag{10}$$

where $\zeta_1 = c_2/(\lambda_1 T)$ and $\zeta_2 = c_2/(\lambda_2 T)$.

Before presenting plots for the thermal contrast of a blackbody (we give these in section 2.3 after the actinometric case has been considered), in order to help gain a better understanding of the thermal contrast as expressed by Eq. (10), a number of important limiting cases will be considered. In the first instance, let $\lambda_1 \to 0^+$ and $\lambda_2 \to \infty$. In these limits $\zeta_1 \to \infty$ and $\zeta_2 \to 0^+$ respectively, so that

$$\lim_{\zeta_1\to\infty}\mathrm{Li}_0(\mathrm{e}^{-\zeta_1}) = \lim_{\zeta_1\to\infty}\frac{\zeta_1^4}{\mathrm{e}^{\zeta_1}-1} = 0 \qquad \text{and} \qquad \lim_{\zeta_2\to 0^+}\mathrm{Li}_0(\mathrm{e}^{-\zeta_2}) = \lim_{\zeta_2\to 0^+}\frac{\zeta_2^4}{\mathrm{e}^{\zeta_2}-1} = 0.$$

Here the zeroth order polylogarithm has been rewritten using its rational form (see Appendix A) followed by repeated application of l'Hôpital's rule to evaluate either limit. Also, since in this limit $\mathfrak{F}_{e,0\to\infty} = 1$ (that is to say it is non-zero), one finally arrives at

$$\mathfrak{T}^{\mathrm{b}}_{e,0\to\infty} = 4. \tag{11}$$

The above limiting value of four is no coincidence and can be readily understood as follows. From the definition for thermal contrast as given by Eq. (6), in this limiting case one has

$$\mathfrak{T}^{\mathrm{b}}_{e,0\to\infty} = \frac{T\dfrac{d}{dT}\displaystyle\int_0^\infty M^{\mathrm{b}}_{e,\lambda}(\lambda,T)\,d\lambda}{\displaystyle\int_0^\infty M^{\mathrm{b}}_{e,\lambda}(\lambda,T)\,d\lambda}. \tag{12}$$

Now as the total radiant exitance for a blackbody radiating into a vacuum is nothing more than the well-known Stefan–Boltzmann law, namely

$$M^{\mathrm{b}}_{\mathrm{e}}(T) = \int_0^\infty M^{\mathrm{b}}_{e,\lambda}(\lambda,T)\,d\lambda = \sigma T^4, \tag{13}$$

where $\sigma = 5.670373\times 10^{-8}$ W m^{-2} K^{-4} is the Stefan–Boltzmann constant, from Eq. (12) it is immediate that

$$\mathfrak{T}^{\mathrm{b}}_{e,0\to\infty} = \frac{T\dfrac{d}{dT}\left(\sigma T^4\right)}{\sigma T^4} = \frac{T(4\sigma T^3)}{\sigma T^4} = 4, \tag{14}$$

as expected.

The limiting value for the thermal contrast found above plays a far more important and significant role in the thermal contrast defined within a given wavelength interval compared to what one may otherwise initially expect. From Eq. (10) one sees that it is also possible for $\mathfrak{T}^{\mathrm{b}}_{e,\lambda_1\to\lambda_2}$ to equal this limiting value for non-zero, finite wavelengths. This occurs when the second term appearing in Eq. (10) is equal to zero. In this case one has

$$\zeta_1^4\mathrm{Li}_0(\mathrm{e}^{-\zeta_1}) = \zeta_2^4\mathrm{Li}_0(\mathrm{e}^{-\zeta_2}), \tag{15}$$

or equivalently

$$\left(\frac{\lambda_1}{\lambda_2}\right)^4 = \frac{\exp\left(\dfrac{c_2}{\lambda_2 T}\right)-1}{\exp\left(\dfrac{c_2}{\lambda_1 T}\right)-1}. \tag{16}$$

For a given wavelength interval, Eq. (16) can be solved for the temperature T. Unfortunately, as Eq. (16) is a transcendental equation in terms of T, it cannot be solved in closed form in terms of the temperature but instead must be found numerically. If the temperature found from solving Eq. (16) numerically is called T_{opt}, the question naturally arises – can any physical significance be attached to the value of the temperature so found? As we now show, it is related to the optimum temperature for maximum efficiency in the production of radiation from a blackbody within a given spectral interval.

As early as 1914 Salpeter,[38] and independently some time later Benfold,[39,40] proposed a measure for the efficiency η of production of radiation from a blackbody source at a selected wavelength can be characterized by

$$\eta_{\text{e},\lambda} = \frac{M^{\text{b}}_{\text{e},\lambda}(\lambda, T)}{M^{\text{b}}_{\text{e}}(T)} = \frac{c_1}{\lambda^5 \left[\exp\left(\frac{c_2}{\lambda T}\right) - 1\right]} \frac{1}{\sigma T^4} = \frac{15 c_2^4}{\pi^4 \lambda^5 \left[\exp\left(\frac{c_2}{\lambda T}\right) - 1\right] T^4}. \tag{17}$$

To find the optimal temperature T_{opt} for maximum spectral efficiency, differentiating Eq. (17) with respect to temperature, setting the result equal to zero and solving for T, the following displacement-like law follows

$$\lambda T_{\text{opt}} = \frac{c_2}{4 + \text{W}_0(-4\text{e}^{-4})} = 3.6697 \text{ mm} \cdot \text{K}. \tag{18}$$

Here $\text{W}_0(x)$ is the Lambert W function (see Appendix B) and is introduced as it allows the displacement-like law for the optimum temperature for maximum spectral efficiency to be written in closed form which otherwise would not be possible.[41]

The spectral efficiency, as has already been noted and is the case when any spectral quantity is so defined, being differential in nature again suffers from a clear, unambiguous physical interpretation being attached to it. Instead the efficiency of production of radiation from a blackbody within a given spectral band has universal meaning and is directly related to what one actually measures experimentally. With this understanding clearly in place, the efficiency of production of radiation from a blackbody within a given spectral band can be readily defined.‡ Here one has

$$\eta_{\text{e},\lambda_1 \to \lambda_2} = \frac{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\text{b}}_{\text{e},\lambda}(\lambda, T) d\lambda}{\displaystyle\int_0^{\infty} M^{\text{b}}_{\text{e},\lambda}(\lambda, T) d\lambda} = \frac{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\text{b}}_{\text{e},\lambda}(\lambda, T) d\lambda}{\sigma T^4}. \tag{19}$$

Of course Eq. (19) is nothing more than the fractional function of the first kind for the radiometric case as given in Eq. (7) but does not appear to have been recognized as such by any of the authors who in the recent past have made used of such a definition for the efficiency of production of blackbody radiation.[56–60] The optimal temperature within a given wavelength interval is found by differentiating Eq. (19) with respect to the temperature T, setting the result equal to zero and solving the resulting equation $\partial \eta_{\text{e},\lambda_1 \to \lambda_2} / \partial T = 0$ for T. Doing so yields Eq. (16) and shows the temperatures at which the thermal contrast is equal to the limiting value of four can be identified with an optimum temperature for maximum efficiency in the production of radiation from a blackbody within a given spectral interval.

As a second limiting case, if one considers $\lambda_1 \to 0^+$, the thermal contrast within the spectral interval from zero up to some finite wavelength $\lambda_2 = \lambda$ for a blackbody can be written as

$$\mathfrak{T}^{\text{b}}_{\text{e},0 \to \lambda T} = 4 + \frac{\zeta^4 \text{Li}_0(\text{e}^{-\zeta})}{\zeta^3 \text{Li}_1(\text{e}^{-\zeta}) + 3\zeta^2 \text{Li}_2(\text{e}^{-\zeta}) + 6\zeta \text{Li}_3(\text{e}^{-\zeta}) + 6 \text{Li}_4(\text{e}^{-\zeta})}. \tag{20}$$

‡Around the turn of the twentieth century it was common practice amongst illumination engineers to speak of the efficiency of a light source as the amount of radiation radiated into the visible portion of the spectrum divided by the total power radiated over all wavelengths.[42–49] This notion of efficiency was first introduced by Langley in 1883[50,51] and was later termed "radiant efficiency" by Nichols in 1894.[52,53] In 1907 Drysdale[54] calculated the percentage of radiation radiated by a blackbody into the visible part of the spectrum using Wien's equation as an approximation to Planck's law while Holladay in the late 1920s used Eq. (19) to find the percentage of radiation radiated by a blackbody into the following four spectral regions: far ultraviolet ($0 - 0.30$ μm), near ultraviolet ($0.30 - 0.40$ μm), visible ($0.40 - 0.76$ μm), and infrared (> 0.76 μm) without the use of any such approximation.[55]

From the series definition for the polylogarithm function with argument $\mathrm{e}^{-\zeta}$, namely

$$\mathrm{Li}_s(\mathrm{e}^{-\zeta}) = \sum_{k=1}^{\infty} \frac{(\mathrm{e}^{-\zeta})^k}{k^s}, \tag{21}$$

as $0 < \zeta < \infty$ one has $0 < \mathrm{e}^{-\zeta} < 1$ and Eq. (21) converges for all $\zeta > 0$. Moreover, $\mathrm{Li}_s(\mathrm{e}^{-\zeta}) > 0$ for all real orders s such that $\zeta > 0$. Thus, as the second term appearing in Eq. (20) consisting of polylogarithms is always positive, one has

$$\mathcal{T}^{\mathrm{b}}_{\mathrm{e},0\to\lambda T} > 4. \tag{22}$$

Moreover, as $\lambda \to \infty$, $\zeta \to 0^+$ and $\mathcal{T}^{\mathrm{b}}_{\mathrm{e},0\to\lambda T} \to 4$, as expected, since it can be readily shown that the second term consisting of polylogarithms in Eq. (20) tends to zero in such a limit. Thus the limiting value of four seen previously for the thermal contrast in a given wavelength interval for a blackbody is a horizontal asymptote approached from above by the curve $\mathcal{T}^{\mathrm{b}}_{\mathrm{e},0\to\lambda T}$.

Finally, the thermal contrast of a blackbody approaches unity regardless of the size of the width of the wavelength band selected in the limit of very large temperatures. To see this, if we write $\zeta = c_2/(\lambda T) = ax$ where $a = c_2/\lambda$ and $x = 1/T$, for large T, x will be small. Noting the following series expansions about the origin

$$\zeta^4 \mathrm{Li}_0(\mathrm{e}^{-\zeta}) = (ax)^4 \mathrm{Li}_0(\mathrm{e}^{-ax}) = a^3x^3 - \frac{a^4x^4}{2} + \frac{a^5x^5}{12} + \mathcal{O}(x^6), \tag{23}$$

and

$$\begin{aligned}\zeta^3\mathrm{Li}_1(\mathrm{e}^{-\zeta}) + 3\zeta^2\mathrm{Li}_2(\mathrm{e}^{-\zeta}) + 6\zeta\mathrm{Li}_3(\mathrm{e}^{-\zeta}) + 6\mathrm{Li}_4(\mathrm{e}^{-\zeta}) &= (ax)^3\,\mathrm{Li}_1(\mathrm{e}^{-ax}) + 3(ax)^2\,\mathrm{Li}_2(\mathrm{e}^{-ax}) \\ &\quad + 6(ax)\,\mathrm{Li}_3(\mathrm{e}^{-ax}) + 6\,\mathrm{Li}_4(\mathrm{e}^{-ax}) \\ &= \frac{\pi^4}{15} - \frac{a^3x^3}{3} + \frac{a^4x^4}{8} - \frac{a^5x^5}{60} + \mathcal{O}(x^6).\end{aligned} \tag{24}$$

Eq. (10) can be rewritten as

$$\mathcal{T}^{\mathrm{b}}_{\mathrm{e},\lambda_1 T\to\lambda_2 T} = 4 + \frac{(\beta^3-\alpha^3)x^3 + \frac{1}{2}(\alpha^4-\beta^4)x^4 + \frac{1}{12}(\beta^5-\alpha^5)x^5 + \mathcal{O}(x^6)}{\frac{1}{3}(\alpha^3-\beta^3)x^3 + \frac{1}{8}(\beta^4-\alpha^4)x^4 - \frac{1}{60}(\alpha^5-\beta^5)x^5 + \mathcal{O}(x^6)} \to 4 - 3 = 1, \tag{25}$$

in the limit of large temperatures, irrespective of the size of the wavelength interval $[\lambda_1, \lambda_2]$. Here $\alpha = c_2/\lambda_1$ and $\beta = c_2/\lambda_2$.

2.2 Actinometric case

For the actinometric case the spectral thermal contrast $\mathcal{T}^{\mathrm{b}}_{\mathrm{q},\lambda}$ is defined in a similar way to what was done for the radiometric case, namely

$$\mathcal{T}^{\mathrm{b}}_{\mathrm{q},\lambda} = \frac{dM^{\mathrm{b}}_{\mathrm{q},\lambda}}{M^{\mathrm{b}}_{\mathrm{q},\lambda}} \Big/ \frac{dT}{T} = \frac{T\left(\dfrac{dM^{\mathrm{b}}_{\mathrm{q},\lambda}}{dT}\right)}{M^{\mathrm{b}}_{\mathrm{q},\lambda}}. \tag{26}$$

Now as the energy carried by a single photon is hc/λ, on dividing Eq. (2) by this amount Planck's law for the spectral photon exitance $M^{\mathrm{b}}_{\mathrm{q},\lambda}$ for a blackbody becomes

$$M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) = \frac{2\pi c}{\lambda^4\left[\exp\left(\dfrac{hc}{k_B\lambda T}\right) - 1\right]}. \tag{27}$$

The suffix "q" is used to denote an actinometric quantity is being considered while the fundamental constants h, c, and k_B are Planck's constant ($6.626\,069\,57 \times 10^{-34}$ J·s), the speed of light in a vacuum ($299\,792\,458$ m s^{-1}), and Boltzmann's constant ($1.380\,6488 \times 10^{-23}$ J·K^{-1}) respectively. It therefore follows the spectral thermal contrast of a blackbody in the actinometric case is

$$\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},\lambda} = \frac{c_2}{\lambda T} \frac{1}{1 - \exp\left(-\dfrac{c_2}{\lambda T}\right)} = \frac{\zeta \, \mathrm{Li}_{-1}(\mathrm{e}^{-\zeta})}{\mathrm{Li}_0(\mathrm{e}^{-\zeta})} = \mathfrak{T}^{\mathrm{b}}_{\mathrm{e},\lambda}, \tag{28}$$

a result identical to the spectral thermal contrast for the radiometric case. This is to be expected as Planck's law for the spectral exitance in either case has the same dependence on temperature.

Next, in a manner similar to the radiometric case, the thermal contrast within the wavelength interval $[\lambda_1, \lambda_2]$ for the actinometric case is defined as

$$\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},\lambda_1 \to \lambda_2} = \frac{T \dfrac{d}{dT} \displaystyle\int_{\lambda_1}^{\lambda_2} M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) \, d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) \, d\lambda}. \tag{29}$$

Once more, the integral appearing in the denominator of Eq. (29) is closely connected to the fractional function of the first kind for the actinometric case defined by

$$\mathfrak{F}_{\mathrm{q},\lambda_1 \to \lambda_2} = \frac{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) d\lambda}{\displaystyle\int_0^{\infty} M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) d\lambda}. \tag{30}$$

After applying Leibniz's rule for differentiating integrals followed by integration by parts the integral appearing in the numerator of Eq. (29) can also be expressed in terms of the fractional function of the first kind for the actinometric case. In closed form in terms of polylogarithms the result is

$$\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},\lambda_1 T \to \lambda_2 T} = 3 + \frac{\zeta_2^3 \, \mathrm{Li}_0(\mathrm{e}^{-\zeta_2}) - \zeta_1^3 \, \mathrm{Li}_0(\mathrm{e}^{-\zeta_1})}{2\zeta(3) \mathfrak{F}_{\mathrm{q},\lambda_1 T \to \lambda_2 T}}. \tag{31}$$

Here $\zeta_1 = c_2/(\lambda_1 T)$, $\zeta_2 = c_2/(\lambda_2 T)$, while $\zeta(x)$ is the Riemann zeta function. As was done previously, we can write

$$\mathfrak{F}_{\mathrm{q},\lambda_1 \to \lambda_2} = \mathfrak{F}_{\mathrm{q},\lambda_1 T \to \lambda_2 T} = \mathfrak{F}_{\mathrm{q},0 \to \lambda_2 T} - \mathfrak{F}_{\mathrm{q},0 \to \lambda_1 T}, \tag{32}$$

where

$$\mathfrak{F}_{\mathrm{q},0 \to \lambda T} = \frac{1}{2\zeta(3)} \left[\zeta^2 \mathrm{Li}_1(\mathrm{e}^{-\zeta}) + 2\zeta \mathrm{Li}_2(\mathrm{e}^{-\zeta}) + 2\mathrm{Li}_3(\mathrm{e}^{-\zeta}) \right]. \tag{33}$$

While not as widely used as the corresponding fractional function of the first kind for the radiometric case, initial study of the function given by Eq. (33) dates back to work first performed in 1931[61] and has a long history of being tabulated.[28, 62–67] Here we have written this fractional function for the very first time in closed form in terms of polylogarithms.[68]

A value for the thermal contrast equal to three in the actinometric case plays an equally important role to the value of four found for the radiometric case. In the limiting case, from the Stefan–Boltzmann law for the total photon exitance one has

$$\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},0 \to \infty} = \frac{T \dfrac{d}{dT} (\sigma_{\mathrm{q}} T^3)}{\sigma_{\mathrm{q}} T^3} = \frac{T (3\sigma_{\mathrm{q}} T^2)}{\sigma_{\mathrm{q}} T^3} = 3, \tag{34}$$

as anticipated. Here σ_{q} ($1.520\,461 \times 10^{15}$ s^{-1} m^{-2} K^{-3}) is the Stefan–Boltzmann constant in photonic units.

Again it is possible for such a limiting value to be obtained for non-zero, finite wavelengths. From Eq. (31) this occurs when

$$\zeta_1^3 \mathrm{Li}_0(\mathrm{e}^{-\zeta_1}) = \zeta_2^3 \mathrm{Li}_0(\mathrm{e}^{-\zeta_2}), \tag{35}$$

which is equivalent to

$$\left(\frac{\lambda_1}{\lambda_2}\right)^3 = \frac{\exp\left(\dfrac{c_2}{\lambda_2 T}\right) - 1}{\exp\left(\dfrac{c_2}{\lambda_1 T}\right) - 1}. \tag{36}$$

Once more the value for T obtained on solving Eq. (36) can be identified with an optimal temperature for maximum efficiency in the production of radiation from a blackbody within a given spectral interval in photonic units.

In the special limiting case where the lower wavelength limit is equal to zero ($\lambda_1 \to 0^+$) the thermal contrast for a blackbody in photonic units can be written as

$$\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},0\to\lambda T} = 3 + \frac{\zeta^3 \mathrm{Li}_0(\mathrm{e}^{-\zeta})}{\zeta^2 \mathrm{Li}_1(\mathrm{e}^{-\zeta}) + 2\zeta \mathrm{Li}_2(\mathrm{e}^{-\zeta}) + 2\mathrm{Li}_3(\mathrm{e}^{-\zeta})}. \tag{37}$$

Using similar arguments to those used for the radiometric case, it can be shown that $\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},0\to\lambda T} > 3$ for all $\zeta > 0$ and the limiting value of three again represents a horizontal asymptote, with the curve of $\mathfrak{T}^{\mathrm{b}}_{\mathrm{q},0\to\lambda T}$ approaching it from above. Lastly, expanding the polylogarithmic terms appearing in Eq. (31) about the origin the thermal contrast of a blackbody for the actinometric case can be shown to approach unity regardless of the size of the width of the wavelength band selected in the limit of large temperatures, a result identical with that found for the radiometric case.

2.3 Results and discussion for the two cases

Thermal contrast isovalues for a blackbody as a function of the wavelength–temperature product for the radiometric (left) and actinometric (right) cases are given in Fig. 1. Qualitatively the behaviour between the two is similar. Isovalues for the thermal contrast between 2 and 8 at steps of 0.5 are given. The wavelengths selected for the wavelength–temperature products correspond to the lower (λ_1) and upper (λ_2) wavelength band limits such that $\lambda_2 > \lambda_1$. In either case the critical curve, shown as dashed, corresponds to isovalues of four (radiometric case) and three (actinometric case) respectively and separates the $\lambda_2 T$–$\lambda_1 T$ plane into two different solution spaces, a feature already predicated by Fang.[21] Each critical curve is governed by Eqs. (16) and (36) respectively, and as was shown in sections 2.1 and 2.2, correspond to limiting values for the thermal contrast obtainable from non-zero, finite wavelengths at temperatures one could identify as an optimal temperature for maximum efficiency in the production of radiation from a blackbody within some given finite spectral interval.

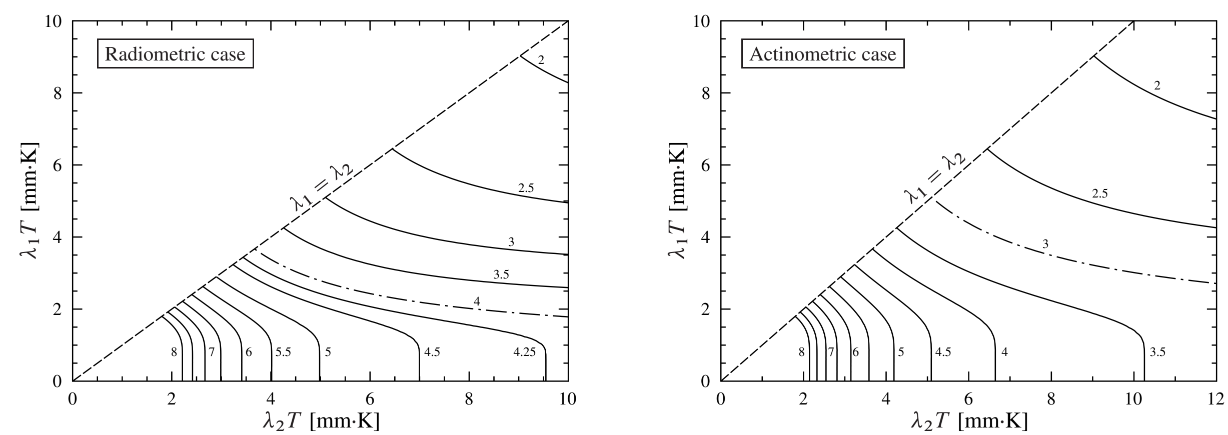

Figure 1: Thermal contrast isovalues for a blackbody as a function of the wavelength–temperature product for wavelengths corresponding to the lower and upper wavelength band limits. LEFT: Radiometric case. RIGHT: Actinometric case. In both cases the dashed curve represents the critical curve and separates the isovalues into two different solution spaces.

Values of the thermal contrast less than the respective values for the critical values, in either case, can only be achieved for spectral bands whose value for the lower wavelength limit is finite. For a lower limiting wavelength value equal to zero

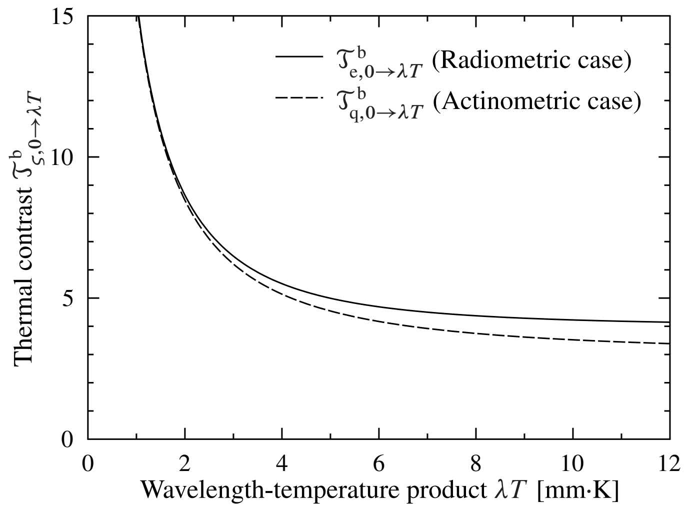

Figure 2: Thermal contrast for a blackbody as a function of the wavelength–temperature product for the wavelength interval $[0, \lambda]$. The solid curve corresponds to the radiometric case ($\varsigma = \mathrm{e}$) while the dashed curved corresponds to the actinometric case ($\varsigma = \mathrm{q}$). For large wavelength–temperature products the limiting values for each curve are three (actinometric) and four (radiometric) respectively.

the thermal contrast approaches its respective critical isovalues from above for increasing wavelength–temperature product and is shown in Fig. 2

In Fig. 3 isothermals for the optimal temperature for maximum efficiency in the production of radiation from a blackbody within some given spectral interval as defined by Eqs (16) and (36) as functions of the lower and upper wavelength bands are given. The radiometric case is to the left while the actinometric case is to the right. Isothermals between 200 K and 1200 K at steps of 100 K are given with the single exception of 250 K. All isothermals shown are at the critical values for the thermal contrast of three for the actinometric case and four for the radiometric case respectively.

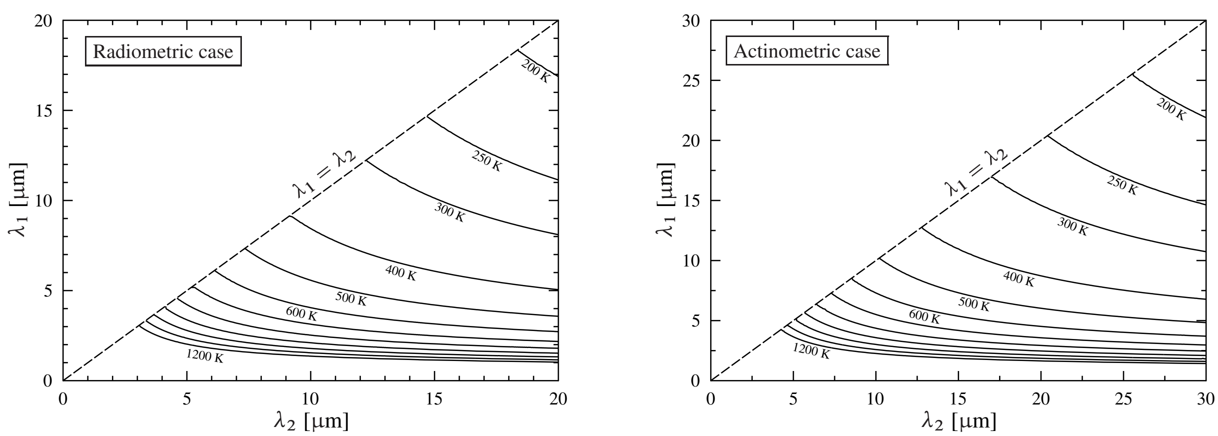

Figure 3: Isothermals for the optimal temperature which gives the maximum efficiency in the production of radiation from a blackbody within some given spectral interval as a function of the lower and upper wavelength band limits. Isothermals between 200 to 1200 K are given in steps of 100 K. LEFT: Radiometric case such that all isothermals correspond to a value of four for the thermal contrast. RIGHT: Actinometric case such that all isothermals correspond to a value of three for the thermal contrast.

It is common for infrared systems to operate in either the mid-wavelength infrared region of 3–5 μm or the long-wavelength infrared region of 8–12 μm. These bands represent two atmospheric windows in the infrared. As an example, in Fig. 4 the thermal contrast of a blackbody in either of these common infrared wavelength intervals as a function of temperature for both the radiometric (left) and actinometric (right) cases are shown. In both cases the 3–5 μm band gives the larger value for the thermal contrast while for the selected bandwidth the size of the thermal contrast between the two cases are very similar. Finally, in both cases the thermal contrast approaches unity in the limit of large temperatures and is independent of the size of the selected wavelength interval.

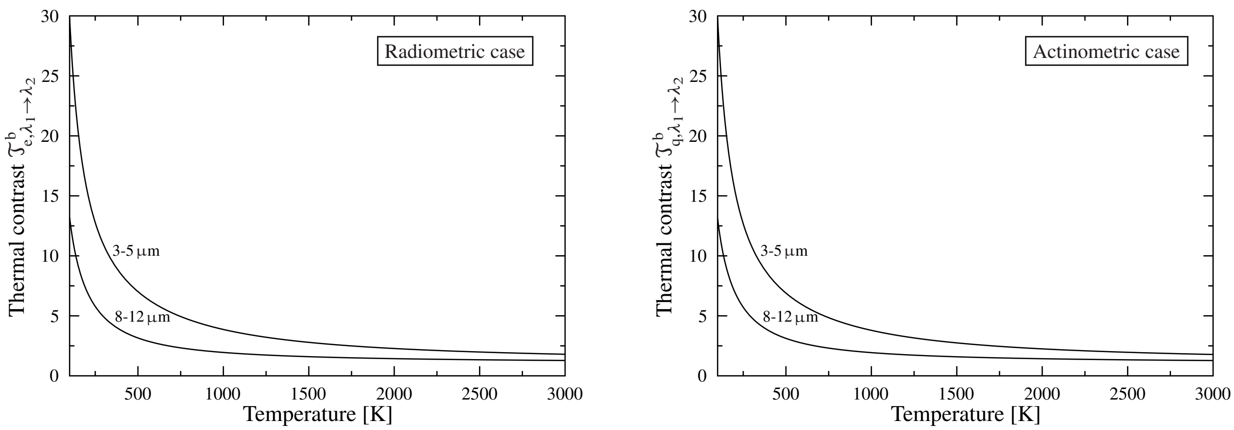

Figure 4: Thermal contrast as a function of the temperature (100–3000 K) for two common wavelength intervals used in the infrared of 3–5 μm (top curve) and 8–12 μm (bottom curve) respectively. LEFT: Radiometric case. RIGHT: Actinometric case.

3. DETECTED THERMAL CONTRAST FOR FOCAL-PLANE RADIATION DETECTING INSTRUMENTS

In this section we develop expressions for the thermal contrast detected by both thermal and quantum detectors for focal-plane radiation detecting instruments. Unlike previous attempts,[8–14, 16–20] in the case where the emissivity for the target can be considered a function of temperature only within some narrow wavelength range, the expressions developed for the detected thermal contrast for both detector types are written in closed form in terms of polylogarithm functions.

Following the work of Scholl and Páez,[8, 9] the spectral irradiance at the image plane of an optical system due to some real body acting as a thermal source can be expressed as

$$E_\lambda(\lambda, T) = \Omega \tau L_{{\rm e},\lambda}(\lambda, T). \tag{38}$$

In this equation τ is the transmission loss factor and will be assumed to be constant over the wavelength interval of interest, Ω is the solid angle subtended by the optical system at the detector plane and is independent of wavelength and temperature,[69, 70] while $L_{{\rm e},\lambda}(\lambda, T)$ is the spectral radiance of the thermal source of interest. For a real body, at its surface its spectral emissivity ε_λ will in general be a function of both its temperature T and its wavelength λ (we assume the surface of the body is optically thick). One can therefore write the spectral radiance for the target body in terms of the spectral radiance for a blackbody $L^{\rm b}_{{\rm e},\lambda}(\lambda, T)$ as follows

$$L_{{\rm e},\lambda}(\lambda, T) = \varepsilon_\lambda(\lambda, T) L^{\rm b}_{{\rm e},\lambda}(\lambda, T), \tag{39}$$

where

$$L^{\rm b}_{{\rm e},\lambda}(\lambda, T) = \frac{c_1}{\pi \lambda^5 \left[\exp\left(\dfrac{c_2}{\lambda T}\right) - 1\right]}. \tag{40}$$

In detecting radiation, the detector is the final component in any system. The quantity one is interested in will therefore be the effective spectral irradiance $E^{\rm D}_{\varsigma,\lambda}(\lambda, T)$ as detected and measured by the sensor in the system. Here the superscript

"D" is used to indicate a quantity related to a detector. As the effective spectral irradiance at the detector is determined by the actual spectral irradiance at the detector weighted by a (dimensionless) spectral response of the detector $\mathcal{R}_\varsigma(\lambda)$, we can write this quantity as

$$E^{\rm D}_{\varsigma,\lambda}(\lambda,T) = \Omega\tau\mathcal{R}_\varsigma(\lambda)L_{{\rm e},\lambda}(\lambda,T) = \Omega\tau\mathcal{R}_\varsigma(\lambda)\varepsilon_\lambda(\lambda,T)L^{\rm b}_{{\rm e},\lambda}(\lambda,T). \tag{41}$$

The form for the weighted spectral response of the detector depends on the underlying physics of how the incident radiation is detected and in general will be a function of wavelength. The generalized subscript ς has been introduced to differentiate between the radiometric case ($\varsigma = {\rm e}$) corresponding to thermal detectors and the actinometric case ($\varsigma = {\rm q}$) corresponding to quantum detectors.

In a manner similar to how the thermal contrast for a blackbody within a given wavelength interval was defined, one can define the detected thermal contrast within some wavelength interval $[\lambda_1, \lambda_2]$ for either thermal or quantum detectors by

$$\mathfrak{T}^{\rm D}_{\varsigma,\lambda_1\to\lambda_2} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2} E^{\rm D}_{\varsigma,\lambda}(\lambda,T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} E^{\rm D}_{\varsigma,\lambda}(\lambda,T)\,d\lambda}. \tag{42}$$

We now find expressions for the detected thermal contrast for either detector type using Eq. (42).

3.1 Ideal thermal detector

For an ideal thermal detector the weighted spectral response can be approximated by a constant which is independent of wavelength, namely $\mathcal{R}_{\rm e}(\lambda) = k$. Using this expression for the weighted spectral response in Eq. (41), substituting into Eq. (42) and simplifying gives

$$\mathfrak{T}^{\rm D}_{{\rm e},\lambda_1\to\lambda_2} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2} \varepsilon_\lambda(\lambda,T)L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} \varepsilon_\lambda(\lambda,T)L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda}. \tag{43}$$

For the simple case were the target can be approximated as a grey body, the spectral emissivity for the body is constant. In this limit one therefore has

$$\mathfrak{T}^{\rm D}_{{\rm e},\lambda_1\to\lambda_2} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2} L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} M^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda} = \mathfrak{T}^{\rm b}_{{\rm e},\lambda_1\to\lambda_2}. \tag{44}$$

Here, since a blackbody is Lambertian (is perfectly diffuse), the well-known relation between the spectral exitance and the spectral radiance of $M^{\rm b}_{\varsigma,\lambda}(\lambda,T) = \pi L^{\rm b}_{\varsigma,\lambda}(\lambda,T)$ has been used. So in the grey body approximation the detected thermal contrast for a thermal detector reduces to the thermal contrast for a blackbody, a purely radiative quantity.

Now for the general case. On applying Leibniz's rule for differentiating integrals Eq. (43) becomes

$$\mathfrak{T}^{\rm D}_{{\rm e},\lambda_1\to\lambda_2} = \frac{T\displaystyle\int_{\lambda_1}^{\lambda_2} \frac{\partial\varepsilon_\lambda}{\partial T}L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda + T\int_{\lambda_1}^{\lambda_2}\varepsilon_\lambda(\lambda,T)\frac{\partial L^{\rm b}_{{\rm e},\lambda}}{\partial T}\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2} \varepsilon_\lambda(\lambda,T)L^{\rm b}_{{\rm e},\lambda}(\lambda,T)\,d\lambda}. \tag{45}$$

Of course, unless an explicit form for the emissivity of the target as a function of both the wavelength and temperature is known it is not possible in general to reduce the above expression for the detected thermal contrast any further.

For cases where the wavelength interval is small, to a good approximation, the spectral emissivity can be assumed to depend only on temperature.[71,72] Within this limit, if $\varepsilon_\lambda(\lambda,T) \simeq \varepsilon(T)$, Eq. (45) can be shown to reduce to

$$\mathfrak{T}^{\rm D}_{{\rm e},\lambda_1\to\lambda_2} = \frac{T}{\varepsilon(T)}\frac{d\varepsilon}{dT} + \mathfrak{T}^{\rm b}_{{\rm e},\lambda_1\to\lambda_2}. \tag{46}$$

The second term appearing in Eq. (46) is nothing more than a radiative term corresponding to the thermal contrast for a blackbody. The first term, on the other hand, is a materially dependent term. As this term can be written as

$$\frac{T}{\varepsilon(T)}\frac{d\varepsilon}{dT} = \frac{d\varepsilon}{\varepsilon}\Big/\frac{dT}{T}\,, \tag{47}$$

it can be interpreted as the ratio of the fractional change in the emissivity to the fractional change in the temperature of the body. For materials whose emissivity increases with increasing temperature this term will be positive and ensures the detected thermal contrast for a thermal detector will always be larger than the thermal contrast for the corresponding blackbody undergoing the same change in temperature. Conversely, for materials whose emissivity decreases with increasing temperature this term becomes negative and leads to the detected thermal contrast for a thermal detector being less than the corresponding thermal contrast for a blackbody. How much larger or smaller the detected thermal contrast will be compared to that of a blackbody source once again depends on the explicit form for the emissivity of the target as a function of temperature. Examples depicting the detected thermal contrast for a target consisting of tungsten will be given in section 3.3.

3.2 Ideal quantum detector

For an ideal quantum detector the weighted spectral response can be approximated by[73]

$$\Re_{\mathrm{q}}(\lambda) = \begin{cases} \dfrac{\lambda}{\lambda_{\max}} & : 0 < \lambda \leqslant \lambda_{\max} \\ 0 & : \lambda > \lambda_{\max} \end{cases} \tag{48}$$

Here $\lambda_{\max}$ is some fixed upper wavelength for which a response in the detector is still possible. Substituting Eq. (48) into Eq. (41) yields

$$E^{\mathrm{D}}_{\mathrm{q},\lambda}(\lambda, T) = \frac{\Omega\tau}{\lambda_{\max}}\varepsilon_\lambda(\lambda, T)\lambda L^{\mathrm{b}}_{\mathrm{e},\lambda}(\lambda, T) = \frac{\Omega\tau}{\lambda_{\max}}\varepsilon_\lambda(\lambda, T)[hcL^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)]. \tag{49}$$

Here $L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)$ is now the spectral photon radiance for a blackbody and is given by

$$L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T) = \frac{2c}{\lambda^4\left[\exp\left(\dfrac{hc}{k_B\lambda T}\right) - 1\right]}. \tag{50}$$

Substituting Eq. (49) into Eq. (42) and simplifying, the detected thermal contrast for an ideal quantum detect can be expressed as

$$\mathfrak{T}^{\mathrm{D}}_{\mathrm{q},\lambda_1\to\lambda_2} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2}\varepsilon_\lambda(\lambda, T)L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2}\varepsilon_\lambda(\lambda, T)L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda}, \qquad \lambda_1 < \lambda_{\max} \leqslant \lambda_2. \tag{51}$$

Leibniz's rule for differentiating an integral can be applied to the numerator of Eq. (51), but as was the case for thermal detectors, unless an explicit form for the emissivity of the target as a function of both its wavelength and temperature are known it is not possible to carry the calculation for the expression of the detected thermal contrast any further.

If the simplifying approximation of a grey body is used for the target, in this limit the detected thermal contrast reduces to

$$\mathfrak{T}^{\mathrm{D}}_{\mathrm{q},\lambda_1\to\lambda_2} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2}L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2}L^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda} = \frac{T\dfrac{d}{dT}\displaystyle\int_{\lambda_1}^{\lambda_2}M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda}{\displaystyle\int_{\lambda_1}^{\lambda_2}M^{\mathrm{b}}_{\mathrm{q},\lambda}(\lambda, T)\,d\lambda} = \mathfrak{T}^{\mathrm{b}}_{\mathrm{q},\lambda_1\to\lambda_2}, \qquad \lambda_1 < \lambda_{\max} \leqslant \lambda_2. \tag{52}$$

The detected thermal contrast for a quantum detector therefore reduces to the thermal contrast for a blackbody in the limit where the grey body approximation applies. Alternatively, if the emissivity is considered to be independent of wavelength over a small spectral band and only a function of the surface's temperature, in this limit the detected thermal contrast for a quantum detector can be shown to be equal to

$$\mathfrak{T}^{\mathrm{D}}_{\mathrm{q},\lambda_1\to\lambda_2} = \frac{T}{\varepsilon(T)}\frac{d\varepsilon}{dT} + \mathfrak{T}^{\mathrm{b}}_{\mathrm{q},\lambda_1\to\lambda_2}, \qquad \lambda_1 < \lambda_{\max} \leqslant \lambda_2. \tag{53}$$

Due to the similarity in forms between Eqs (46) and (53), similar comments as to those made for the thermal case apply to the quantum case.

3.3 Results and discussion

Different types of detectors lead naturally enough to different properties in the detected thermal contrast. The detected thermal contrast depends intimately on the functional form of the weighted spectral response of the detecting instrument used. In the limit where the emissivity can be taken to depend only on temperature within a narrow wavelength band, from Eqs (46) and (53) one sees contributions to the detected thermal contrast arise as a result of changes in the value for the emissivity over the object's surface and from a radiative term resulting from changes in its temperature.

Compared to the previous work of others, where a number of approximations were developed and used, the expressions we give for the detected thermal contrast for either detector type are exact. For example, Páez and Scholl in their extensive work on this problem[8, 9, 11–17, 19, 20] only ever gave expressions for the detected thermal contrast in terms of truncated series approximations they termed a *compensated* approximation.[74–76] Their approximation was compensated in the sense the number of terms used in the truncated series is adjusted according to the size of an error term with the error term being included in the series summation. Fang,[10] on the other hand, obtained results for the detected thermal contrast using a Gauss–Laguerre approximation to the infinite series which appeared in his expressions[33, 34] and claimed his expressions were simpler compared to those given by Páez and Scholl since they made it possible for greater physical information to be more readily extracted from his equations. This is not something we would entirely agree with. Instead we claim the polylogarithmic formulation developed here is simpler and by far the most elegant of all the formulations promulgated so far for calculating the detected thermal contrast. Our exact expressions also provide a clearer, more intuitive physical picture, as they show how and from where each term appearing in the expression for the detected thermal contrast arises.

As already alluded to, for many real surfaces over a very narrow wavelength range the emissivity will be independent of wavelength and can be well approximated by a linear temperature dependent function of the form

$$\varepsilon(T) \simeq a_0 + a_1(T - T_0). \tag{54}$$

Here a_0 and a_1 are constants while T_0 is taken to be room temperature, that is $T_0 = 300\ K$. In the well-known grey body approximation $a_1 = 0$ and $0 < a_0 < 1$ while in the linear approximation a_0 and a_1 are non-zero. Within the linear approximation the term appearing in the detected thermal contrast due to a fractional change in the emissivity to the fractional change in the temperature (see Eq. (47)) can be expressed as

$$\frac{T}{\varepsilon(T)}\frac{d\varepsilon}{dT} = \frac{1}{1 + \frac{1}{T}\left(\frac{a_0}{a_1} - T_0\right)}. \tag{55}$$

In an absolute sense (since the term may be positive or negative depending on the values for the constants a_0 and a_1) this term is a positive, monotonically increasing function bounded between zero and one for $T > 0$. So within a linear approximation for the emissivity with respect to temperature, the fractional change in the emissivity resulting from a fractional change in temperature can only contribute a value to the detected thermal contrast for either detector type that is no greater than (or less than) one (or minus one).

In the literature extensive data for the spectral emissivity of tungsten has been reported at various temperatures and over a range of wavelength intervals.[77–82] Applying a linear least-squares fit to tungsten emissivity data reported by Aksyutov[82] at a wavelength of 4 μm and for temperatures ranging from 0 to 600°C, one finds

$$\varepsilon(T) = 0.032 + 7.028 \times 10^{-5}(T - T_0), \quad 273 \leqslant T \leqslant 873\ \mathrm{K}. \tag{56}$$

In Fig. (5) the detected thermal contrast as a function of temperature for a body whose emissivity is linearly dependent on temperature (solid curve) and for a body whose emissivity is approximated using the grey body approximation (dashed curve) for the two common infrared wavelength intervals of 3–5 μm and 8–12 μm are shown for the case of a thermal (left) and quantum (right) detector. Within a linearly temperature dependent model for the emissivity it can be seen that for either detector type the 3–5 μm band gives the larger value for the detected thermal contrast while the values for the detected thermal contrast within either band considered are very similar for each detector type. Both these observations are in direct contrast to what has been previously reported by Aranda, Páez and Scholl.[20] They find the detected thermal contrast for either detector type is higher in the 8–12 μm wavelength interval compared to the 3–5 μm interval. They also report the size of the detected thermal contrast is significantly larger for a quantum detector compared to a thermal detector, exhibiting up to 20 times better performance for the former compared to the latter. We, on the other hand, do not find this. Instead, within the linearly temperature dependent emissivity model used, the performance between either detector type are equally matched as their detected thermal contrasts at equal temperatures are approximately the same.

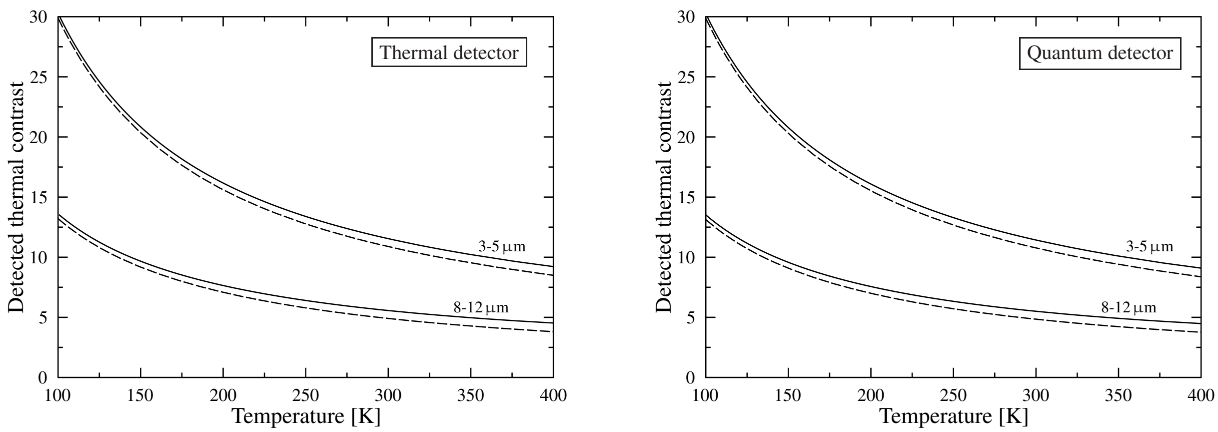

Figure 5: Detected thermal contrast as a function of the temperature from a real surface for two common wavelength intervals used in the infrared of 3–5 μm (top two curves) and 8–12 μm (bottom two curves) respectively. Solid curve is for a body within a linear temperature dependent emissivity model. Dashed curve is for a body within the grey body approximation. LEFT: Thermal detector. RIGHT: Quantum detector.

Aranda, Páez and Scholl also seem to overstate the contribution to the detected thermal contrast arising from the inclusion of a temperature dependent emissivity for the target. Without its inclusion, they claim the detected thermal contrast will be in error by an amount approximately equal to 30% when operating at room temperatures. We on the other hand find the contribution arising from the inclusion of a temperature dependent emissivity term to be somewhat less. As an example, consider a temperature of 300 K (room temperature). We find the percentage increase in the detected thermal contrast by its inclusion compared to the case where the grey body approximation is used are approximately 6 and 13.5% for the bands 3–5 μm and 8–12 μm respectively and is irrespective of the type of detector. Of course within such a linearly temperature dependent emissivity model the percentage increase in the detected thermal contrast will change depending on the temperature selected. However, as already noted, the maximum contribution arising from the emissivity dependent term is unity and occurs in the limit of large temperatures. In the grey body approximation the detected thermal contrast has a limiting value of one for large temperatures, a value which is independent of detector type and the size of the selected wavelength band. Within a linear temperature dependent model for the emissivity the greatest possible increase in the detected thermal contrast by its inclusion is at most twice as large, a result only achievable at temperatures well beyond those of normal room temperatures.

4. CONCLUSION

We have developed exact, closed form expressions for the thermal contrast detected by thermal and quantum detectors for focal-plane radiation detecting instruments in terms of polylogarithm functions valid in the limit where the emissivity can be assumed to be dependent on temperature over very narrow wavelength intervals of interest. The formulation employed allowed the various contributions to the detected thermal contrast to be readily seen. Here, for either detector type, two terms contribute to the detected thermal contrast. The first was a material dependent emissivity term while the second was a purely radiative term. Examples are given for targets made of tungsten using a simple model for the emissivity which is linearly dependent on temperature. The work not only extends a number of approximate methods that have been used in the past to calculate the detected thermal contrast but also shows that differences between detector types in the value for the detected thermal contrast is not as significant as has been previously suggested by others in the literature.

APPENDIX A. POLYLOGARITHMS

The polylogarithm function, or polylogarithms for short, is one of the classical special functions of mathematical physics known for over 250 years.[83] While known for many years, only in the last few decades has one seen a sudden resurgence in interest and use of the function. Defined by the series[84]

$$\mathrm{Li}_s(x) = \sum_{k=1}^{\infty} \frac{x^k}{k^s} \quad |x| < 1, \tag{57}$$

by the process of analytic continuation, for fixed order s, its domain can be extended to all real x where it is defined.

The polylogarithms are so named from a close connection they have with the natural logarithm. When $s = 1$, from Eq. (57) it is apparent

$$\mathrm{Li}_1(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} = -\ln(1-x), \tag{58}$$

and is just the Mercator series for the natural logarithm with x replaced by $-x$. In this way the polylogarithms can be thought of as one of the simplest generalizations of the logarithm.

Of particular interest to the work presented here are polylogarithms with arguments of the form e^{-x}. For such arguments, derivatives and integrals for the polylogarithms reduce to simple ladder operations on the order index together with an associated change in sign. Here differentiating the polylogarithm with respect to x lowers its order by one, namely

$$\frac{d}{dx}\mathrm{Li}_s(\mathrm{e}^{-x}) = -\mathrm{Li}_{s-1}(\mathrm{e}^{-x}), \tag{59}$$

while integrating the polylogarithm, to within an arbitrary constant, raises its order by one, namely

$$\int \mathrm{Li}_s(\mathrm{e}^{-x})\, dx = -\mathrm{Li}_{s+1}(\mathrm{e}^{-x}). \tag{60}$$

Polylogarithms of zero and negative integer orders turn out to be rational functions of their argument.[85] The first few are $\mathrm{Li}_0(x) = x/(1-x)$, $\mathrm{Li}_{-1}(x) = x/(1-x)^2$, $\mathrm{Li}_{-2}(x) = x(1+x)/(1-x)^3$, etc. Finally it should be noted that as the polylogarithm function is implemented in many computer algebra systems, for example `polylog[s,x]` in Maple and `PolyLog[s,x]` in Mathematica, numerical values for the function can be readily found.

APPENDIX B. THE LAMBERT W FUNCTION

The Lambert W function,[84] denoted by $\mathrm{W}(x)$, is defined to be the inverse of the function $f(x) = x\mathrm{e}^x$ satisfying

$$\mathrm{W}(x)\,\mathrm{e}^{\mathrm{W}(x)} = x. \tag{61}$$

Referred to as the defining equation for the Lambert W function, Eq. (61) is multivalued and therefore has an infinite number of branches of which only two are real. By convention, the branch satisfying $\mathrm{W}(x) \geqslant -1$ is taken to be the principal branch, denoted by $\mathrm{W}_0(x)$, while that satisfying $\mathrm{W}(x) < -1$ is known as the secondary real branch and is denoted by $\mathrm{W}_{-1}(x)$.

For real arguments Eq. (61) can have either one unique positive real root $\mathrm{W}_0(x)$ if $x \geqslant 0$ except for $\mathrm{W}_0(0) = 0$; two negative real roots $\mathrm{W}_0(x)$ and $\mathrm{W}_{-1}(x)$ if $-1/\mathrm{e} < x < 0$; one negative real root $\mathrm{W}_0(-1/\mathrm{e}) = \mathrm{W}_{-1}(-1/\mathrm{e}) = -1$ if $x = -1/\mathrm{e}$; and if $x < -1/\mathrm{e}$ there are no real roots. The branch point between the two real branches ($k = 0, -1$) occurs at $x = -\mathrm{e}^{-1}$.

The function is named in honour of Johann Heinrich Lambert (1728–77) who in 1758 was the first to consider a problem requiring $\mathrm{W}(x)$ for its solution. The function, however, would not be formally named in the literature for a further 230 years.[86] Its immense popularity and use is a modern phenomenon driven largely by its recent inclusion in computer algebra systems such as Maple's `LambertW[k,x]`, Sage's `lambert_w[k,x]`, and Mathematica's `ProductLog[k,x]`.

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