SENSITIVITY COEFFICIENTS ANALYSIS

Svetozár Malinarič[1], Peter Ďuríšek[2]

[1]Department of Physics, Faculty of Natural Sciences, Constantine the Philosopher University, Tr. A. Hlinku 1, SK-949 74 Nitra, Slovakia
[2]ON Semiconductor Slovakia, Vrbovská cesta 2617/102, SK-92101 Piešťany, Slovakia
Email: smalinaric@ukf.sk, peter.durisek@onsemi.com

Abstract

The contribution deals with thermophysical parameter estimation in dynamic methods. The influence of temperature measurement uncertainty on the parameter estimation uncertainty is studied using least squares procedure. Difference analysis is used for time window determination in which the fitting procedure should be applied. The sensitivity coefficient analysis is performed on Pulse transient, Step-wise transient and EDPS method.

Key words: parameter estimation, least squares procedure, uncertainty, differential analysis

1 Introduction

The methods used for measuring thermophysical parameters of materials can be divided into steady state and dynamic ones. While the former use a steady state temperature field inside the sample, the latter use a dynamic temperature field. The dynamic methods [1] can be characterized as follows. The temperature of the sample is stabilized and uniform. Then the dynamic heat flow in the form of a pulse or step-wise function is applied to the sample. The thermophysical parameters of the material can be calculated from the temperature response

The measuring procedure consists of theory and experiment. The theoretical model of the experiment is described by the partial differential equation for the heat transport. The temperature function is a solution of this equation with boundary and initial conditions corresponding to the experimental arrangement. The experiment consists in measuring the temperature response and fitting the temperature function over the experimental points. Using the least squares procedure following thermophysical parameters can be estimated: thermal diffusivity a, thermal conductivity λ and specific heat capacity c.

2 Least squares procedure

As mentioned above the first step of evaluation is to determine the temperature function - temperature increase as a function of time. Assume the function is of known analytic form

$$f(t,\boldsymbol{\alpha}) = f(t,\alpha_1,\alpha_2,...\alpha_p) \qquad (1)$$

where t is a variable and $\boldsymbol{\alpha}$ is a vector of unknown parameters [2]. In addition to one or two thermophysical parameters, there are usually some nuisance parameters connected with the model [3]. We suppose that the deviation between model and experiment is negligible and the only source of uncertainty, in this analysis, stems from temperature measuring accuracy. We also assume that the uncertainties of temperature measurement of all points are the same and uncertainties of time measurement are negligible. As the temperature function (1) is nonlinear in parameters we have to expand it using Taylor series [4]. Then we can use the linear least squares procedure in matrix notation

$$\mathbf{T} - f(\mathbf{t}, \mathbf{a}) = \mathbf{X} \cdot (\boldsymbol{\alpha} - \mathbf{a}) + \boldsymbol{\varepsilon} \tag{2}$$

where $\mathbf{T}$ is a vector of temperature measured at n points determined by $\mathbf{t}$ vector of times. $\boldsymbol{\varepsilon}$ is a vector of errors, $\mathbf{a}$ is a close guess for parameter vector $\boldsymbol{\alpha}$ and $\mathbf{X}$ is a sensitivity matrix [2] given by

$$\{X_{ij}\} = \frac{\partial f(t_i, \boldsymbol{\alpha})}{\partial \alpha_j} = \beta_j(t_i, \boldsymbol{\alpha}) \tag{3}$$

where

$$\beta_j(t, \boldsymbol{\alpha}) = \frac{\partial f(t, \boldsymbol{\alpha})}{\partial \alpha_j} \tag{4}$$

is a sensitivity coefficient for parameter α_j. The sensitivity coefficient is a measure of the change in temperature function due to the variation of the estimated parameter. Then the least squares estimate $\mathbf{a}_{LS}$ of the parameter vector $\boldsymbol{\alpha}$ is given by the form

$$\mathbf{a}_{LS} - \mathbf{a} = \left(\mathbf{X}^T \cdot \mathbf{X}\right)^{-1} \cdot \mathbf{X}^T \cdot (\mathbf{T} - f(\mathbf{t}, \mathbf{a})) \ . \tag{5}$$

The standard uncertainty of the parameter α_j estimate becomes

$$u(\alpha_j) = A_j u(T) \tag{6}$$

where

$$A_j^2 = \left\{\left(\mathbf{X}^T \cdot \mathbf{X}\right)^{-1}\right\}_{jj} \tag{7}$$

and $u(T)$ is a standard uncertainty of temperature measurement. If T is regarded as input quantity and α_j as output one then A_j can be also considered as sensitivity coefficient defined by GUM [5]. So the parameter estimation uncertainty consists of two parts. The first is given by temperature function and selection of measured points. The second is given by temperature measurement uncertainty.

3 Difference analysis

In this section we will focus on optimizing the experiment with respect to the data window defined by the time interval $(t_B, t_B + t_S)$, where t_B is the beginning and t_S the size of the interval, as shown in Figure 1. The difference analysis [6] is a method for the time window determination in which the fitting procedure should be applied to obtain reliable values of thermophysical parameters. The method is based on estimating parameters using least squares procedure when t_B is successively changed while t_S is kept constant. The results of fitting are plotted against t_B. If the time interval $(t_B, t_B + t_S)$ is not suitable for parameter estimation, the results of fitting are erroneous and the plot is scattered.

The difference analysis can be applied to data from real measurements where all types of uncertainties are included. It can also be used in experiment modelling where the only source of uncertainty is simulated as random noise of temperature measurement. The third application of difference analysis consist in plotting the time dependence of coefficient A_j. As seen from the equation (6), low value of A_j predicts also low value of parameter uncertainty $u(\alpha_j)$ and thus low scattering of parameter values α_j computed using least squares estimation. Described procedure has been used for time interval determination in three dynamic methods of thermophysical parameters measurement.

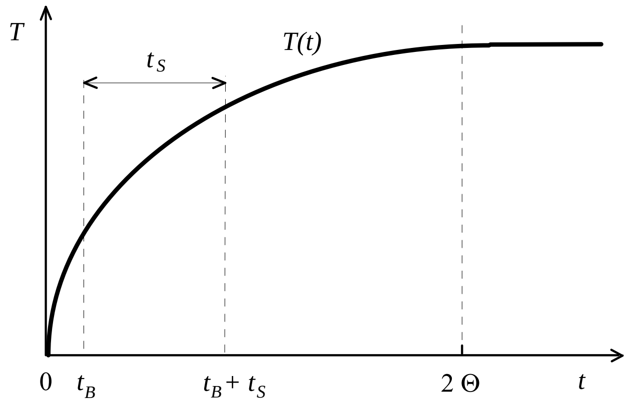

Fig 1 Temperature function and data window definition

4 Pulse transient method

This method is characterized by pulse heating and one-dimensional heat flow into a infinite sample. The temperature is measured at a distance *h* from the plane heat source. The temperature function [1] is given by

$$T(t,a,c) = \frac{Q}{c\rho\sqrt{\pi a \ t}} \exp\left(-\frac{h^2}{4at} \right) \qquad (7)$$

where *a* is the thermal diffusivity, *c* is the specific heat capacity, *Q* is the heat energy of pulse and ρ is density. Figure 2 shows the temperature function and the sensitivity coefficients β_a and β_c as a function of time. Nondimensional time scale is defined by the Fourier number $F = at/h^2$. Figure 3 shows the time dependence of coefficients A_a and A_c. The window in which the fitting procedure should be applied is $0.3 < F < 0.6$ for specific heat and $0.05 < F < 0.3$ for thermal diffusivity.

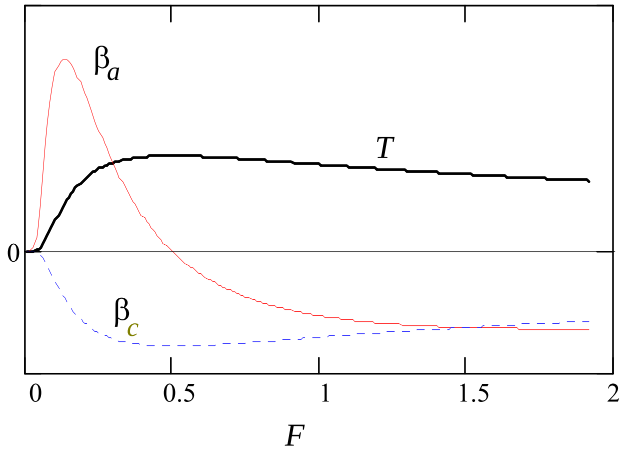

Fig 2 Temperature function and sensitivity coefficients β_a and β_c vs nondimensional time scale

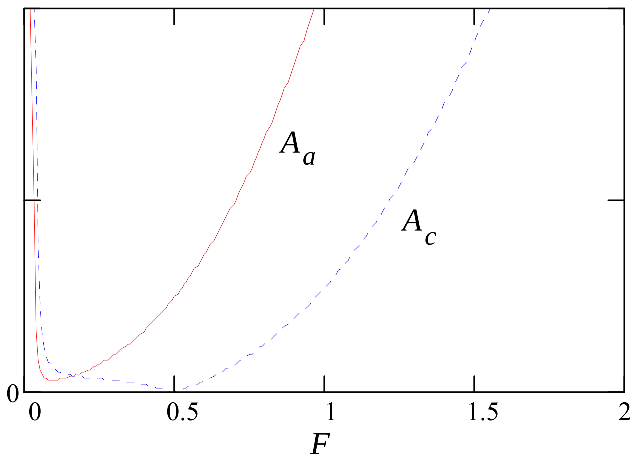

Fig 3 Values of coefficients A_a and A_c vs nondimensional time scale

5 Step-wise transient method

This method is characterized by step-wise heating and one-dimensional heat flow into a infinite sample. The temperature is measured at a distance h from the plane heat source. The temperature function [7] is given by

$$T(t,a,c)=\frac{q}{ac\rho}\left[\sqrt{\frac{at}{\pi}}\exp\left(-\frac{h^2}{4at}\right)-\frac{h}{2}\operatorname{erfc}\left(\frac{h}{\sqrt{4at}}\right)\right] \tag{8}$$

where q is the heat current density and erfc is the error function integral [8]. Figure 4 shows the temperature function and the sensitivity coefficients β_a and β_c as a function of time. Figure 5 shows the time dependence of coefficients A_a and A_c.The window in which the fitting procedure should be applied is $0.8 < F < 1.8$ for specific heat and $0.1 < F < 1.5$ for thermal diffusivity.

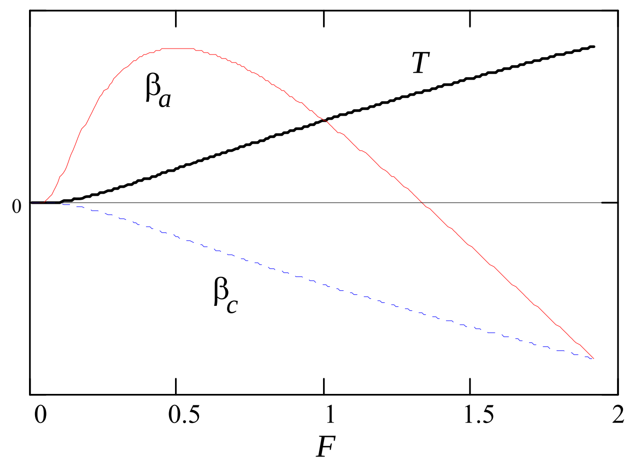

Fig 4 Temperature function and sensitivity coefficients β_a and β_c vs nondimensional time scale

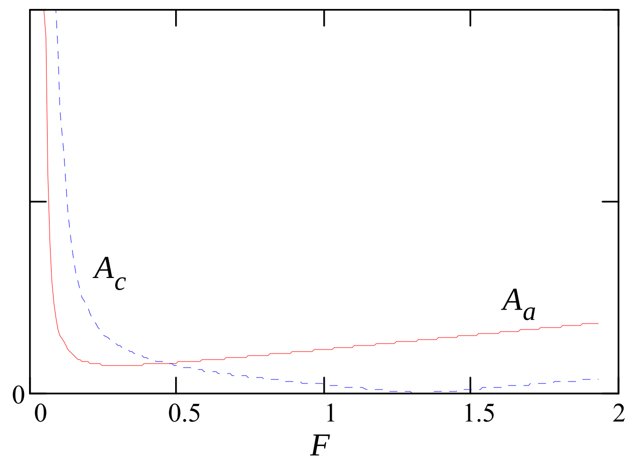

Fig 5 Values of coefficients A_a and A_c vs nondimensional time scale

6 Extended dynamic plane source (EDPS) method

This method is characterized by step-wise heating and one-dimensional heat flow into a finite sample at the isothermal boundary conditions. The nickel disc simultaneously serves as the heat source and thermometer. The temperature function [9] is given by

$$T(t,a,c,\tau)=\frac{q}{ac\rho}\sqrt{\frac{at}{\pi}}\left[1+2\sqrt{\pi}\sum_{n=1}^{\infty}\beta^{n}\,\mathrm{ierfc}\left(\frac{nh}{\sqrt{at}}\right)\right]+\tau \tag{9}$$

where q is the heat current density, h is the specimen thickness and ierfc is the error function integral [8]. Nuisance parameter τ is the base line referred to the additional increase in the temperature of the disc due to its imperfections. Figure 6 shows the temperature function and the sensitivity coefficients β_a and β_c as a function of time. Figure 7 shows the time dependence of coefficients A_a and A_c. The window in which the fitting procedure should be applied is $0.1 < F < 0.8$ for specific heat and $0.1 < F < 1.1$ for thermal diffusivity.

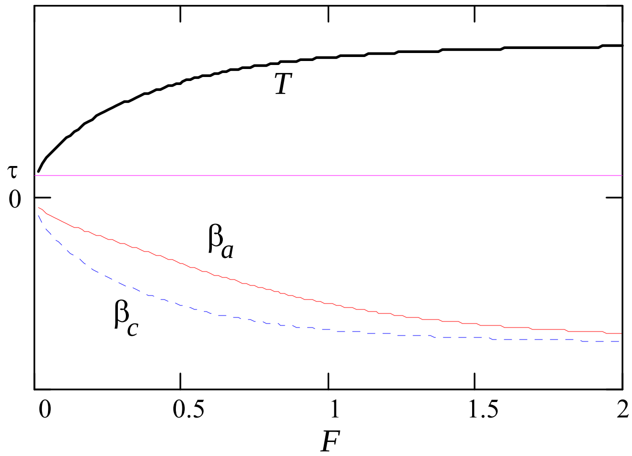

Fig 6 Temperature function and sensitivity coefficients β_a and β_c vs nondimensional time scale

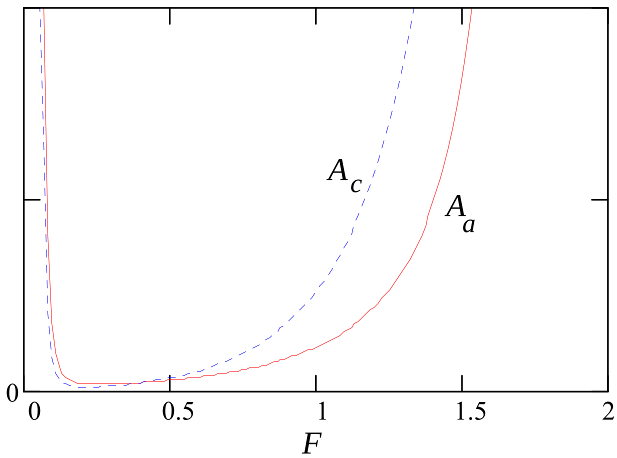

Fig 7 Values of coefficients A_a and A_c vs nondimensional time scale

7 Conclusions

The work showed close connection between difference analysis and sensitivity coefficients analysis. Both methods solve the same question: How long the temperature response should be scanned and what time interval should be used for fitting. But there is a significant difference between these methods. While in difference analysis we need particular points with experimental or simulated noise, in sensitivity coefficients analysis we need only the temperature function. Presented analysis consists in plotting the time dependence of coefficient A defined by equation (7). The proper time interval has been assessed where coefficient A attained relatively low values. The analysis was performed on Pulse transient (Fig 3), Step-wise transient (Fig 5) and EDPS (Fig 7) method. The results are in good agreement with those of other methods [10, 11, 12].

Acknowledgement

Authors wish to thank the Slovak Science Grant Agency for the financial support under the contract 1/0279/03.

References

[1] Kubičár Ľ and Boháč V, 1999 *Proc. 24th Int. Conf. on Thermal Conductivity / 12th Int. Thermal Expansion Symp.* (October 26-29, 1997) ed P S Gaal and D E Apostolescu (Lancester: Technomic) p.135

[2] Beck J V and Arnold K J, 1977 *Parameter Estimation in Engineering and Science* (New York: Wiley)

[3] Vozár L, Groboth G, 1997 *High Temp.-High Press.* **29**, 191

[4] Kundracik F, 1999 *Spracovanie experimentálnych dát* (Bratislava, UK)

[5] ISO 1993 *Guide to the Expression of the Uncertainty in Measurement* (Geneva : ISO)

[6] Boháč V, Gustavsson M K, Kubičár Ľ and Gustafsson S E, 2000 *Rev. Sci. Instrum.* **71**, 2452

[7] Kubičár Ľ and Boháč, 2000 *Meas. Sci. Technol.* **11,** 252

[8] Carslaw H S and Jaeger J C, 1959 *Conduction of Heat in Solids* (Oxford: Clarendon)

[9] Karawacki E, Suleiman B M, ul-Hag I and Nhi B, 1992 *Rev. Sci. Instrum.* **63**, 4390

[10] Kubičár Ľ and Boháč, 2000 *Thermophysics 2000* Proc. of the Meeting of the Thermophysical Society - Working Group of the Slovak Physical Society (Nitra: CPU) p. 39

[11] Malinarič S, 2003 *Thermophysics 2003* Proc. of the Meeting of the Thermophysical Society - Working Group of the Slovak Physical Society (Nitra: CPU)

[12] Malinarič S, 2004 *Meas. Sci. Technol.* **14**, 807