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BY

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ANALYSIS OF ERRORS DUE TO EDGE HEAT LOSS IN GUARDED HOT PLATES*

BY WILLIAM WOODSIDE[1]

SYNOPSIS

A concise, easily evaluated analytical expression for the error in thermal conductivity measurement by the guarded hot plate, due to heat exchange with the ambient air, is derived assuming that the temperature distribution along the specimen edges may be represented by a mean temperature. This solution agrees closely with one presented by Somers and Cyphers (**1**)[2] for the special case of the specimen edges held at the cold plate temperature.

The error is shown to depend upon three dimensionless parameters: (*a*) the ratio of guard ring width to specimen thickness; (*b*) the ratio of the length of side of the test area to specimen thickness; and (*c*) a number between zero and unity whose value is determined by the specimen-edge-temperature distribution. The ASTM specimen thickness requirements[3] are based only upon the first parameter. The approximate effect of the size of the test area upon the error is described.

A procedure is proposed for testing, when necessary, very thick specimens. This involves the measurement of the specimen-edge-temperature distribution during test (with or without edge insulation), calculation of the error, and correction of the measured conductivity.

The guarded hot plate method for determining the thermal conductivity of building and insulating materials assumes unidirectional heat conduction through the test specimens in the region of the central test area. This is equivalent to assuming that the isothermals in the section of the specimens adjacent to the test area of the heater plate are planes parallel to the hot and cold plate surfaces. The accuracy of the method depends upon the validity of this assumption.

Distortion of the isothermals may be caused by small differences in temperature or unbalances between the test and guard areas of the heater plate. Errors resulting from this source are discussed in a companion paper (**2**). However, even when the test area and guard ring are perfectly balanced, unidirec-

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[2] The boldface numbers in parentheses refer to the list of references appended to this paper, see p. 62.

[3] Method of Test for Thermal Conductivity of Materials by Means of the Guarded Hot Plate (C 177 – 45), 1955 Book of ASTM Standards, Part 3, p. 1084.

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tional heat flow may not occur in the test section of the specimens if the width of the guard ring is insufficient for the thickness of specimen being tested. The edges of the test specimens and the edge of the guard ring lose or gain heat by exchange with the ambient air. This results in a non-linear temperature distribution along the edge of the specimen even when edge insulation is applied to reduce the magnitude of the edge heat losses. The resulting distortion

TABLE I.—ASTM SPECIMEN THICKNESS REQUIREMENTS.

Maximum Thickness of Test Specimens, l_{max}, in.	Minimum Linear Hot Plate Dimensions, in., square or circular		g/l_{max}
	Central Test Area, $2s$[a]	Guard Ring Width, g	
1	4	1½	1½
1½	8	2¼	1½
2	12	3	1½
4	12	6	1½

[a] Length of side of test area, in.

of the isothermals and heat flow lines may extend into the test section of the specimens and produce an error in the measured conductivity if the specimen thickness is too large or the guard ring width too small.

For this reason, maximum permissible specimen thicknesses for given heater plate dimensions have been specified by the ASTM[3] and other organizations in their standard methods of test. Table I shows the ASTM maximum specimen thickness requirements. From this table, it is evident that these requirements are based upon a lower limiting value of 1.5 for the ratio of guard ring width, g, to specimen thickness, l. The thickness requirements of RILEM[4, 5, 6] and the British Standards Institution (BSI)[7] are based on values of 1.5 and 1.0 respectively for the lower limit of the ratio g/l. However the bases on which these minimum values of g/l were selected are not clear. The ASTM standard further specifies, apparently arbitrarily, minimum linear dimensions of test area for various thicknesses of test specimens. In addition to restricting specimen thicknesses, the standard requires that the edges of the heater plate, test specimens, and cold plates be thermally insulated to reduce the net edge heat loss or gain, with edge insulation having a thermal resistance at least twice that of the specimen. There does not appear to be any published information which gives the basis for these requirements.

PREVIOUS WORK

In 1951, Somers and Cyphers (1) presented a rigorous analytical solution which permitted the error in measured conductivity to be calculated from the known heater plate dimensions and the thickness of the specimens being tested. This solution assumed that the surface heat transfer coefficient h on the edges of the specimens approaches infinity ($h \rightarrow \infty$), and that the ambient air temperature equals the temperature of the cold plates. This was equivalent to assuming that the specimen edges are maintained at a fixed temperature equal

[4] Réunion Internationale des Laboratoires D'Essais et de Recherches sur les Matériaux et les Constructions—International Assn. of Testing and Research Labs. for Mats. and Structures.

[5] *Bulletin*, International Assn. of Testing and Research Labs. for Mats. and Structures, No. 19, pp. 1–12 (1954).

[6] Method of Test for Thermal Conductivity of Building Materials by Means of the Guarded Hot Plate, Laboratorio Nacional de Engenharia Civil, Ministerio das Obras Publicas, Portugal.

[7] Definition of Heat Insulating Terms and Methods of Determining Thermal Conductivity and Solar Reflectivity, *Standard No. 874*, British Standards Institution (1939).

to the cold plate temperature, which, of course, is not usually the case, and which results in much larger distortions of the isothermals and heat flow lines in the specimens than would normally occur. The errors predicted by Somers and Cyphers were thus much larger than the actual errors, but how much larger could not be determined. Furthermore, their solution involves the summation of a double infinite series that does not converge rapidly and therefore is not simple to evaluate. They concluded that the maximum permissible thicknesses specified by ASTM could be more than doubled if an error of 5 per cent, as calculated from their equation, were accepted.

Dusinberre (3) in 1952 used the relaxation technique to predict errors due to edge heat loss. By assuming a more realistic value of h/k[8] than did Somers and Cyphers, and by applying the relaxation method to one specific set of test conditions (given values of g, s, and l, ambient temperature and temperature difference between hot and cold plates) he obtained an error equal to one half of that predicted by Somers and Cyphers for the same conditions of g, s, and l.

More recently Pascal (4) has investigated this problem. He determined, by the relaxation method, the minimum value of the ratio g/l, guard ring width to specimen thickness, which would retain unidirectional heat flow in the test section of the specimens. The temperature distribution along the edges of specimens was obtained by measurement on actual specimens which were well insulated against edge heat loss to the ambient air. The result of Pascal's relaxation solution is shown in Fig. 1 with the isothermals and heat flow lines. Beyond the dotted line, the isothermals are substantially parallel to the hot- and cold-plate surfaces. The dotted line repre-

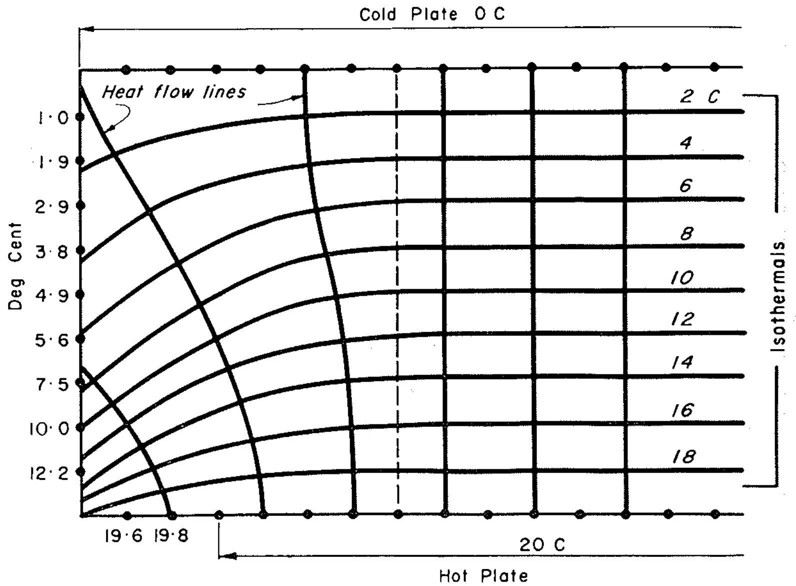

FIG. 1.—Pascal's Measured Specimen-Edge-Temperature Distribution for a Thick Specimen Well Insulated Around its Edges, and the Heat Flow Lines and Isothermals Obtained by the Relaxation Method.

[8] h = surface heat transfer coefficient for specimen or edge insulation, Btu per hr sq ft deg Fahr.
k = thermal conductivity of test specimens, Btu in. per hr sq ft deg Fahr.

sents a value of $g/l = 0.7$ in contrast to the minimum permissible value specified by ASTM, $g/l = 1.5$. Two criticisms may be made of this solution. It was assumed that the only parameter affecting edge heat loss errors was the ratio g/l, whereas, as will be shown later, the ratio s/l, the ratio of the test area dimension to specimen thickness, is also an important parameter. Secondly, in performing the relaxation, two-dimensional flow was assumed. This assumption is incorrect since the area for lateral heat conduction but which is more readily adaptable to realistic boundary conditions.

The following assumptions are made:

1. The hot and cold plate surfaces are isothermal. This assumption neglects the existence of the gap separating the central test area and the guard ring of the heater plate, and also presupposes that the test and guard areas are maintained at the same temperature, that is, that the plate is perfectly "balanced."

2. The edges of specimens are maintained at a uniform temperature, this increases in proportion to the distance from the center of the test area.

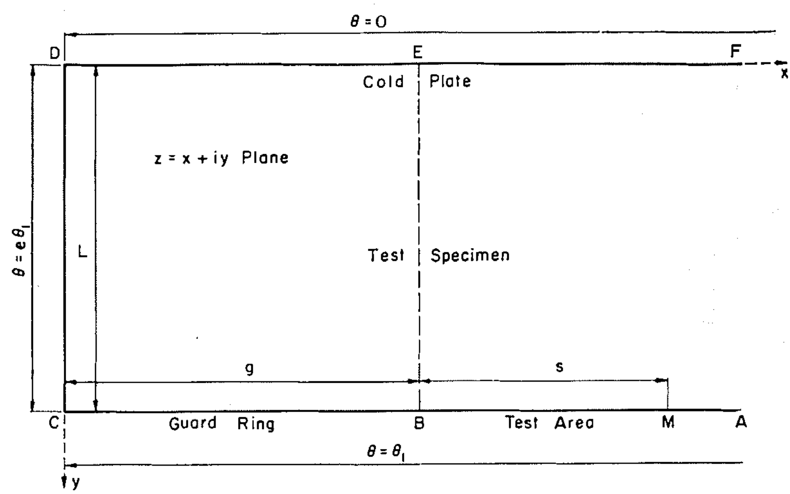

FIG. 2.—Diagram Representing Part of the Cross-Section Through the Hot Plate - Test-Specimen - Cold Plate System at the Center Plane, Showing the Coordinate Axes and Boundary Conditions.

THEORETICAL ANALYSIS

It is thus apparent that the magnitude of errors in measured conductivity due to edge heat loss is not known with any degree of precision. In this paper a theoretical solution to the problem is developed which agrees with that of Somers and Cyphers for the special boundary conditions assumed by them

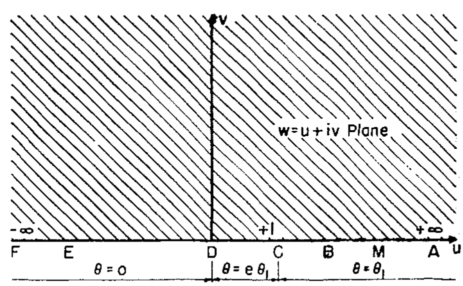

FIG. 3.—The w-Plane into Which the z-Plane is Transformed.

temperature being between the hot and cold plate temperatures. Thus if the cold plate temperature is taken as $\theta = 0$ and the hot plate temperature as $\theta = \theta_1$, then the edges of the specimen are assumed to be uniformly at a fixed temperature $\theta = e\theta_1$, where $0 \leqslant e \leqslant 1$. The solution obtained will therefore depend upon the value of e selected. Putting $e = 0$ gives the case of the edges being maintained at the cold plate temperature, which was the assumption made by Somers and Cyphers. Thus the solution derived here will contain that of Somers and Cyphers as a special case.

Figure 2 represents part of a cross-section through the hot plate - test specimen - cold plate system at the center plane. M represents the center of the test area, BC the guard ring width, and CD the specimen edge for a square heater plate. The cold plate DEF is at temperature $\theta = 0$, the specimen edge CD at temperature $\theta = e\theta_1$, and the hot plate $AMBC$ at temperature $\theta = \theta_1$. D is chosen to be the origin of a Cartesian coordinate system.

To determine the two-dimensional temperature distribution in the specimen under the above boundary conditions, Fig. 2 will be transformed by means of a Schwarz-Christoffel conformal transformation to Fig. 3 where the temperature distribution is more easily determined. This distribution will then be transformed back to Fig. 2 using the inverse transformation. In Fig. 2, A and F may be taken to be at $x = +\infty$, since this will not affect the temperature distribution in the region to the left of M $(x \leqslant x_M)$ which is the region of interest.

The Schwarz-Christoffel transformation which transforms the w-plane $(v \geqslant 0)$ to the z-plane ($AMBCDEF$ and the space enclosed) is given by:

$$\frac{dz}{dw} = C(w - 0)^{(\pi/2\pi)-1}(w - 1)^{(\pi/2\pi)-1}$$

$$= \frac{C}{\{w(w - 1)\}^{\frac{1}{2}}}$$

where C is a constant to be determined from the known boundary conditions.

Integrating,

$$z = C\{\cosh^{-1}(2w - 1) + C'\}$$

where C' is also a constant.

When $z = 0$, $w = 0$; therefore $C' = -i\pi$.

Also when $z = il$, $w = +1$.

$$\therefore\ il = -i\pi C,\ C = -l/\pi, \quad \text{and}$$

$$z = -l/\pi\{\cosh^{-1}(2w - 1) - i\pi\}$$

Transposing,

$$\cosh^{-1}(2w - 1) = i\pi - \pi z/l$$

or,

$$w = \frac{1}{2}(1 - \cosh \pi z/l) \qquad (1)$$

This is the transformation that transforms Fig. 2 to Fig. 3. Replacing w by $(u + iv)$ and z by $(x + iy)$ and separating real and imaginary components, Eq 1 becomes:

$$u = \frac{1}{2}\left(1 - \cosh\frac{\pi x}{l}\cdot\cos\frac{\pi y}{l}\right),$$
$$v = -\frac{1}{2}\left(\sinh\frac{\pi x}{l}\cdot\sin\frac{\pi y}{l}\right). \quad \ldots\ldots (2)$$

The problem is now to determine the temperature distribution in Fig. 3 above the u-axis, with AC at temperature $\theta = \theta_1$, CD at temperature $\theta = e\theta_1$, and DF at temperature $\theta = 0$.

According to Carslaw and Jaeger (5), in the half-plane $v \geqslant 0$ with $v = 0$ maintained at temperature $f(u)$, the steady-state temperature distribution is given by:

$$\theta(u, v) = \frac{v}{\pi}\int_{-\infty}^{+\infty}\frac{f(u')\,du'}{v^2 + (u - u')^2} \quad \ldots\ldots (3)$$

This result will now be applied to the w-plane in Fig. 3. In this figure the following relations hold:

$$f(u') = 0 \quad \text{for} \quad -\infty < u' < 0,$$
$$f(u') = e\theta_1 \quad \text{for} \quad 0 < u' < 1,$$
$$f(u') = \theta_1 \quad \text{for} \quad 1 < u' < +\infty.$$

$$\therefore\ \theta(u, v) = \frac{v}{\pi}e\theta_1\int_0^1\frac{du'}{v^2 + (u - u')^2} + \frac{v}{\pi}\theta_1\int_1^{\infty}\frac{du'}{v^2 + (u - u')^2}$$
$$= \frac{\theta_1}{\pi}\left\{e\tan^{-1}\left(\frac{u}{v}\right) + (1 - e)\tan^{-1}\left(\frac{u - 1}{v}\right) + \frac{\pi}{2}\right\}.$$

This is the temperature distribution in the w-plane. To obtain the distribution in the z-plane, u and v are simply replaced by the values given by Eq 2.

$$\therefore\ \theta(x, y) = \frac{\theta_1}{2} + \frac{\theta_1}{\pi}\left\{e\tan^{-1}\left(\frac{\cosh\frac{\pi x}{l}\cdot\cos\frac{\pi y}{l} - 1}{\sinh\frac{\pi x}{l}\cdot\sin\frac{\pi y}{l}}\right) + (1 - e)\tan^{-1}\left(\frac{\cosh\frac{\pi x}{l}\cdot\cos\frac{\pi y}{l} + 1}{\sinh\frac{\pi x}{l}\cdot\sin\frac{\pi y}{l}}\right)\right\} \quad \ldots\ldots (4)$$

By differentiating Eq 4 with respect to y and then putting $y = l$, the temperature gradient at the hot plate surface is found to be:

$$\left(\frac{\partial\theta}{\partial y}\right)_{y=l} = -\frac{\theta_1}{l}\left\{e\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l} + 1}\right) + (1 - e)\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l} - 1}\right)\right\} \quad \ldots\ldots (5)$$

Therefore the heat flux out of the test area is:

$$q = -k\left(\frac{\partial\theta}{\partial y}\right)_{y=l} = \frac{k\theta_1}{l}\left\{c\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l}+1}\right) + (1-e)\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l}-1}\right)\right\},$$

where k is the thermal conductivity of the specimens. Under ideal conditions when there are no edge effects and no distortion of the isothermals and heat flow lines, the heat flux would be simply $k\theta_1/l$ which is independent of x. This expression is modified by the factor in braces which takes account of the distortion of the ideal temperature distribution by edge effects.

This factor has been derived for the two-dimensional case, that is, for a system having the same cross-section as is shown in Fig. 2 but extending to infinity in the two directions normal to the plane of the figure.

The product solution giving the variation of the heat flux q over the hot plate surface of a square heater plate, that is, the three-dimensional case, is as follows:

$$q = \frac{k\theta_1}{l}\left\{e\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l}+1}\right) + (1-e)\left(\frac{\sinh\frac{\pi x}{l}}{\cosh\frac{\pi x}{l}-1}\right)\right\} \times \left\{e\left(\frac{\sinh\frac{\pi x'}{l}}{\cosh\frac{\pi x'}{l}+1}\right) + (1-e)\left(\frac{\sinh\frac{\pi x'}{l}}{\cosh\frac{\pi x'}{l}-1}\right)\right\}, \quad \ldots (6)$$

where x' is the space variable corresponding to x in a direction normal to the plane of Fig. 2.

The total heat flow out of one face of the test area will therefore be:

$$Q = 4\int_{x=g}^{g+s}\int_{x'=g}^{g+s} q\cdot dx\cdot dx'.$$

Performing the integration the following result is obtained:

$$Q = \frac{4k\theta_1 l}{\pi^2}\left\{e\ln\left(\frac{\cosh\frac{\pi(g+s)}{l}+1}{\cosh\frac{\pi g}{l}+1}\right) + (1-e)\ln\left(\frac{\cosh\frac{\pi(g+s)}{l}-1}{\cosh\frac{\pi g}{l}-1}\right)\right\}^2 \quad \ldots\ldots\ldots\ldots (7)$$

But if k_{exp} is the experimentally measured thermal conductivity under these boundary conditions, Q is also given by:

$$Q = k_{exp}\,(2s)^2\frac{\theta_1}{l} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

By equating these two expressions for Q, given by Eqs 7 and 8, the relationship between the "true" conductivity k and the experimentally determined conductivity

k_{exp} is found to be:

$$\left(\frac{k}{k_{\text{exp}}}\right)^{\frac{1}{2}} = \frac{\pi s/l}{e \ln\left(\dfrac{\cosh \dfrac{\pi(g+s)}{l} + 1}{\cosh \dfrac{\pi g}{l} + 1}\right) + (1-e)\ln\left(\dfrac{\cosh \dfrac{\pi(g+s)}{l} - 1}{\cosh \dfrac{\pi g}{l} - 1}\right)} \quad \ldots\ldots (9)$$

This expression from which the error ($k_{\text{exp}}/k - 1$) in thermal conductivity due to edge heat losses may be calculated is, as would be expected, independent of the conductivity of the specimens being tested. The error depends only upon the two dimensionless parameters s/l and g/l and also upon e which in turn depends upon the edge temperature distribution. Also it satisfies the basic requirement that the error is zero for large values of g/l, for if g is large compared with l, then:

$$\cosh \frac{\pi(g+s)}{l} = \frac{1}{2}\exp\left\{\frac{\pi(g+s)}{l}\right\} \quad \text{and} \quad \cosh\frac{\pi g}{l} = \frac{1}{2}\exp\left(\frac{\pi g}{l}\right).$$

$$\therefore \left(\frac{k}{k_{\text{exp}}}\right)^{\frac{1}{2}} = \frac{\pi s/l}{e \ln\left(\exp\dfrac{\pi s}{l}\right) + (1-e)\ln\left(\exp\dfrac{\pi s}{l}\right)} = 1.$$

Thus the error predicted by Eq 9 does tend toward zero as g/l increases.

Several special cases having different specimen-edge-temperature distributions are presented:

Case I: $e = 0$

This is the case treated by Somers and Cyphers, where edges of the specimen are maintained at cold plate temperature.

When $e = 0$ is substituted into Eq 9 the following expression is obtained:

$$\left(\frac{k}{k_{\text{exp}}}\right)^{\frac{1}{2}} = \frac{\pi s/l}{\ln\left(\dfrac{\cosh \pi(g+s)/l - 1}{\cosh \pi g/l - 1}\right)} \quad \ldots\ldots (10)$$

This is to be compared with the expression of Somers and Cyphers which is given in terms of the two dimensionless parameters:

$$y_1/y_0 = s/(g+s) \quad \text{and} \quad z_0/y_0 = l/(g+s)$$

and is

$$\frac{k}{k_{\text{exp}}} = \frac{\dfrac{\pi^3}{64}\left(\dfrac{y_1}{y_0}\right)^2\left(\dfrac{2y_0}{z_0}\right)}{\displaystyle\sum_{m,n=0}^{\infty} \frac{(a^2+b^2)^{\frac{1}{2}}}{a^2b^2}\cdot\coth\left\{(a^2+b^2)^{\frac{1}{2}}\frac{\pi z_0}{2y_0}\right\}\sin\left(a\cdot\frac{\pi y_1}{2y_0}\right)\sin\left(b\,\frac{\pi y_1}{2y_0}\right)}, \quad \ldots\ldots (11)$$

where $a = (2m+1)$ and $b = (2n+1)$.

Somers and Cyphers presented their results graphically, plotting k/k_{exp} *versus* z_0/y_0 for two values of the parameter y_1/y_0, namely 0.5 and 0.8. The evaluation of k/k_{exp} for any other value of y_1/y_0 from their results would entail either a very rough estimate from their plotted results or further calculation on the basis of Eq 11.

The agreement between Eqs 10 and 11 is very close as is evident from Fig. 4 where the results of Somers and Cyphers are compared with those calculated from Eq 10 for the appropriate values of s/l and g/l.

Case II: $e = 1$

In this case, the edges of the specimens are maintained at the hot plate temperature. For this special case, Eq 9 becomes:

$$\left(\frac{k}{k_{exp}}\right)^{\frac{1}{2}} = \frac{\pi s/l}{\ln\left(\dfrac{\cosh \pi(g + s)/l + 1}{\cosh \pi g/l + 1}\right)} \quad \text{(12)}$$

The errors calculated from this equation are similar in magnitude but of course opposite in sign ($k > k_{exp}$) to those calculated from Eq 10.

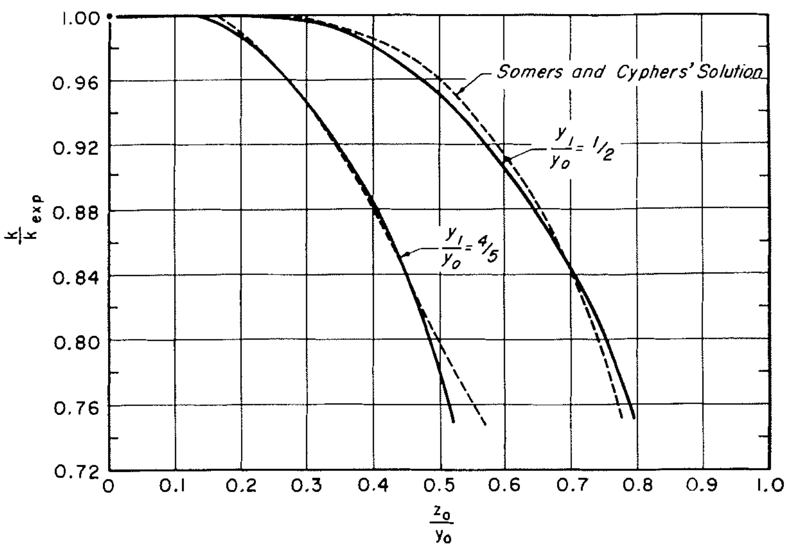

FIG. 4.—Comparison Between the Results of Somers and Cyphers and those Calculated from Eq 10.

Case III: $e = 0.5$

This case assumes the specimen edges to be maintained at a temperature equal to the mean of the hot and cold plate temperatures. Equation 9 now becomes:

$$\left(\frac{k}{k_{exp}}\right)^{\frac{1}{2}} = \frac{\pi s/l}{\ln\left(\dfrac{\sinh \pi(g + s)/l}{\sinh \pi g/l}\right)} \quad \text{(13)}$$

This equation predicts relatively small errors, since the boundary condition $e = 0.5$ is approximately representative of the ideal case where the temperature gradient along the edges of the specimen would be linear, as in the central part of the specimen.

Equation 9 predicts accurately the error in measured conductivity for the special cases of the edges of specimens maintained at any uniform temperature between the hot and cold plate temperatures. In the following section, a method is proposed for the representation of actual specimen-edge-temperature distributions by a mean constant edge temperature and a corresponding value of e.

DISCUSSION

Figure 5 shows k/k_{exp} calculated from Eq 9 plotted against specimen thickness l for an 8 by 8-in. hot plate having $s = g = 2$ in., for different values of the parameter $e = 0$, 0.29, 0.35, 0.40, 0.45, and 0.50. The value $e = 0.29$ was calculated from the edge-temperature distribution (shown in Fig. 1) measured by Pascal on a thick specimen well insulated around its edges. The method used in arriving at this value of e is described later. This value of e applies only to the specimen thickness, edge insulation, temperature conditions, etc., in the particular test performed by Pascal, but is plotted in Fig. 5 since it does represent an actual measured edge temperature distribution. Figure 6 is a similar graph of k/k_{exp} *versus* specimen thickness for a

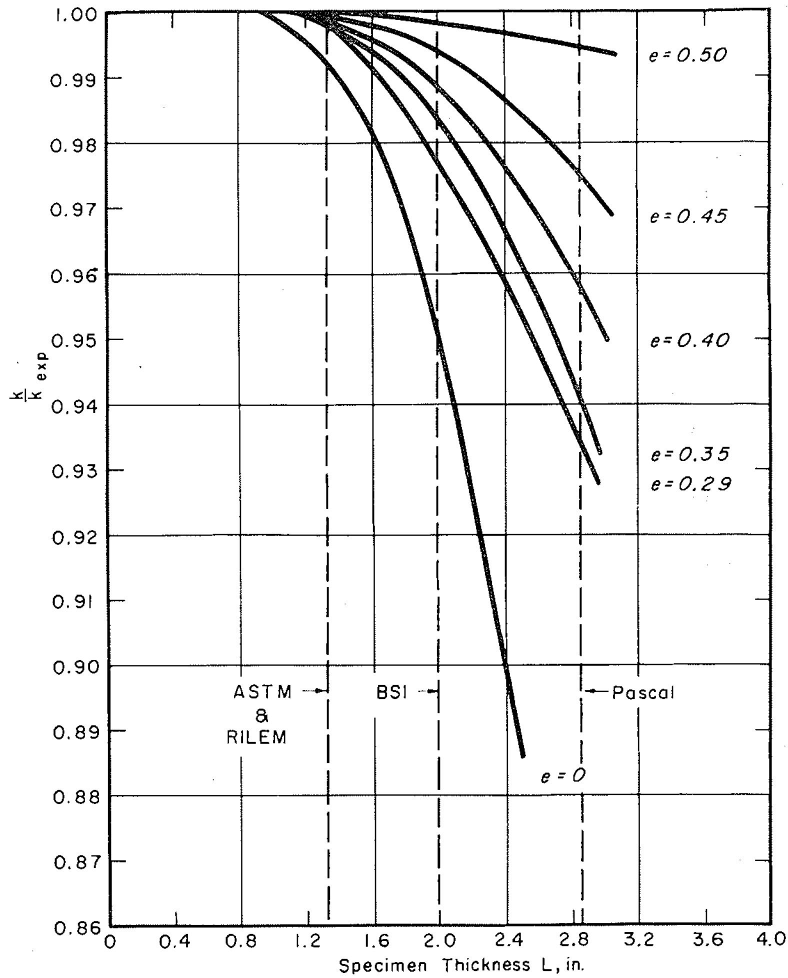

FIG. 5.—k/k_{exp} Calculated from Eq 9 Plotted Against Specimen Thickness for an 8 by 8-in. Hot Plate Having $s = g = 2$ in. for Different Values of e.

hot plate 18 by 18 in. having $s = 6$, $g = 3$ in., for values of $e = 0, 0.35, 0.40, 0.45$, and 0.50.

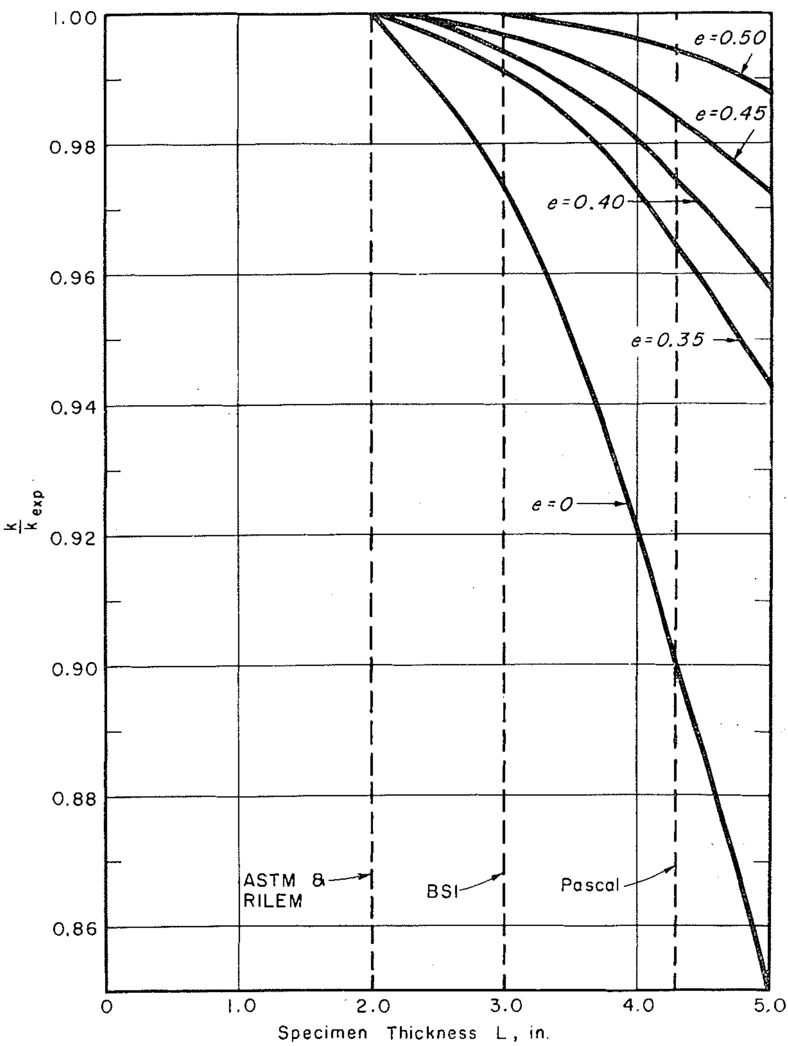

FIG. 6.—k/k_{exp} Calculated from Eq 9 Plotted Against Specimen Thickness for an 18 by 18-in. Hot Plate Having $s = 6$, $g = 3$ in. for Different Values of e.

ductivity. Curve c represents the average or mean temperature of the specimen edge for the linear distribution curve a

In Fig. 7, which shows several hypothetical specimen-edge-temperature distributions, curve a represents the ideal linear temperature distribution along the edge of the specimen from which no error would result in the measured conductivity. Curve c represents the average or mean temperature of the specimen edge for the linear distribution curve a and is approximated by a value for e of 0.5 (Eq 13). The calculated errors for $e = 0.5$ are non-zero for large specimen thicknesses (see Figs. 5 and 6), but the deviation from zero, which is the true error, is small. For example, in Fig. 5 for

$e = 0.5$ and l as high as 3 in., the calculated error is only 0.6 per cent.

This suggests that the actual measured specimen-edge-temperature distribution may be approximately represented by an average edge temperature for the purposes of error evaluation; this average edge temperature defines a corresponding value of e, and it appears that the error calculated from Eq 9 with this value of e will differ only slightly from the true error. Suppose the meas-

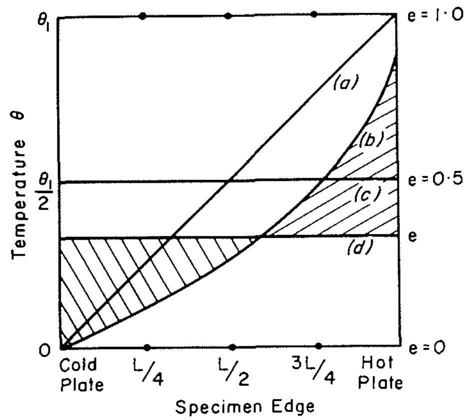

FIG. 7.—Several Hypothetical Specimen-Edge-Temperature Distributions.

ured specimen-edge-temperature distribution is represented by curve b in Fig. 7. The mean edge temperature is then given by curve d, which is drawn in such a way that the shaded areas are equal. This defines an approximate value of e for such a distribution. Then using this value of e, together with the known values of g, s, and l for the test under consideration, the approximate error to be expected in the measured conductivity may be calculated. This approximate error will be larger than the true error, since the assumption of a constant edge temperature produces a still greater distortion of the heat flow lines. However, the difference between the true and calculated errors may only be an amount less than or equal to the amount by which the calculated $e = 0.5$ error deviates from zero, which has already been shown to be small. This will require experimental verification.

Since the specimen-edge-temperature distribution depends upon the specimen thickness, the corresponding value of e will also depend upon thickness. This means that an experimentally determined curve of k/k_{exp} *versus* specimen thickness will not coincide with any of the calculated error curves obtained by assuming a constant value of e, such as those shown in Figs. 5 and 6.

A difficulty in measuring the specimen-edge-temperature distribution may be pointed out. If the hot plate apparatus is a horizontal one, that is with the plates and specimens stacked one above the other, the edge temperature distribution will be the same on all four edges of the specimen. If, however, the apparatus is arranged vertically, which is the most common arrangement, different edge temperature distributions will occur on the top, side, and bottom edges of the specimens. Such an arrangement would require the measurement of the edge distributions on all four edges, the average then being taken for the calculation of e.

As has been pointed out, both ASTM and RILEM require $g/l_{max} = 1.5$, whereas the BSI permits $g/l_{max} = 1.0$. Pascal from his work concluded that $g/l_{max} = 0.7$ would result in negligible errors when the edges of specimens are well insulated. For the heater plate considered in Fig. 5, the maximum permissible thicknesses for test specimens according to these three requirements would be 1.33, 2.00, and 2.86 in. respectively. These are shown in Fig. 5, where it may be seen that, if e is assumed to be 0.29, specimens of these thicknesses would result in errors of approximately 0.2, 2.3 and 6.6 per cent, respectively.

Figure 5 suggests that the ASTM maximum specimen thickness requirement is not too stringent for accurate test work with an 8 by 8-in. hot plate ($s = g = 2$ in.). Comparing Fig. 5 with Fig. 6, which is for an 18 by 18-in. hot plate, the error curves in Fig. 6 are displaced to the right (that is, the direction of increasing thickness) with respect to the ASTM maximum thickness limit. This suggests that thicker specimens than are allowed by the ASTM requirement might be tested in larger plates with the same accuracy, allowing the value of g/l_{max} to decrease as the plate size increases. At present the ASTM requirement specifies minimum linear dimensions of the test area for a given test specimen thickness (see Table I). A requirement which specified values of g/l_{max} for different sizes of test areas would permit more flexibility in the design of heater plates than does the present requirement.

TABLE II.—EFFECT OF SIZE OF TEST AREA UPON ERROR IN MEASURED CONDUCTIVITY.

$e = 0.29$. $g = l = 2$ in.

Where e is a dimensionless number representative of specimen edge temperature distribution.

Length of Side of Heater Plate, $2(g + s)$, in.	Half-Length of Side of Test Area, s, in.	Per Cent Error in Measured Conductivity $(1 - k/k_{exp}) \times 100$
6	1	3.75
8	2	2.30
12	4	1.19
16	6	0.80

The effect of the size of the test area upon the error in measured conductivity is illustrated in Table II, where, for a guard ring width of 2 in., a specimen thickness of 2 in., and an edge-temperature distribution represented by $e = 0.29$, the percentage error in measured conductivity, $(1 - k/k_{exp}) \times 100$, as calculated from Eq 9, is shown for heater plates having different sizes of test area.

To determine the permissible variation of the ratio g/l_{max} with size of test area $2s$, to keep errors in measured conductivity below any desired limiting value (say 0.25 per cent), a series of experimental tests with different sizes of hot plate is required. If the difference between the true error and that calculated from Eq 9 is found to be small, then to determine the error for any specific set of hot plate dimensions, all that would be required would be the measurement of the specimen-edge-temperature distribution under the test conditions. Such a series of tests is being planned in this laboratory with the three available sizes of hot plates.

To reduce errors caused by edge heat loss to a minimum, the ASTM standard method of test requires that insulation be placed around the edges of the heater plate, test specimens, and cold plates, "of such a thickness that the resistance to edge losses shall be at least twice and preferably three or more times the thermal resistance of the specimen in the direction of normal heat flow." There is no published information that provides the basis for this requirement. Also it is a blanket requirement for all tests, taking no account of varying test conditions. It would be more satisfactory if the exact amount of edge insulation could be specified for any given set of test conditions. The more edge insulation applied, the closer will the specimen-edge-temperature distribution tend toward the ideal straight-line distribution for which there is no error. The error depends, however, upon other factors besides the thermal resistance to edge losses. In the case of no edge insulation, the specimen-edge-temperature distribution, and hence the error, depend upon the conductivity, thickness, and surface heat transfer coefficient of the specimens, the ambient air temperature, the mean temperature and the hot plate - cold plate temperature difference of the test. When edge insulation is applied, the specimen-edge-temperature distribution and the error depend upon the thickness and conductivity of the edge insulation, as well as upon all the above factors. The orientation of the apparatus, whether vertical or horizontal, will also be a factor for both cases. A limited testing program is being set up to determine the amount of edge insulation required for several test conditions, and, in this test series too, the analytical solution presented may prove to be of assistance.

The situation often arises where specimens must be tested whose thickness exceeds the maximum permissible in the available hot-plate apparatus. In such a situation the error may be minimized by applying large amounts of edge insulation and keeping the ambient air temperature close to the mean temperature of test. Even when these precautions have been taken, the error in testing thick specimens may be large. If the specimen-edge temperature is measured during test, the appropriate value of e may be calculated, the approximate error determined from Eq 9, and the measured conductivity corrected accordingly.

CONCLUSIONS

The following conclusions may be drawn from the analysis presented:

1. An analytical expression for the error in conductivity measurement by the guarded hot plate, due to edge heat exchange with the ambient air, has been obtained, assuming that the actual specimen-edge-temperature distribution may be represented by a uniform mean temperature. The solution agrees closely with that of Somers and Cyphers for the boundary conditions assumed by them and is much simpler to evaluate.

2. The ratio of the length of side of test area of the heater plate to the specimen thickness is an important factor in determining the error, as well as the ratio of guard ring width to specimen thickness. It is shown that for a prescribed guard ring width and acceptable error, the larger the test area, the thicker the specimens that may be tested.

3. Experimental data are needed to enable the establishment of maximum specimen thickness requirements and edge insulation requirements on a more rational basis. The analytical solution presented may perhaps be of assistance in this respect. Experimental data are also needed to test the applicability of the procedure suggested for testing very thick specimens, when necessary.

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